

Evaluation of double and triple integration:

① $\int_0^1 \int_x^{\sqrt{x}} (x^2 + y^2) dy dx$

→ let $I = \int_{x=0}^1 \int_{y=x}^{\sqrt{x}} (x^2 + y^2) dy dx$

$$= \int_{x=0}^1 \left(x^2 \cdot y + \frac{y^3}{3} \right)_{y=x}^{\sqrt{x}} dx$$

$$= \int_{x=0}^1 \left[x^2 (\sqrt{x} - x) + \frac{1}{3} ((\sqrt{x})^3 - x^3) \right] dx$$

$$= \int_{x=0}^1 \left[x^{5/2} - x^3 + \frac{x^{3/2}}{3} - \frac{x^3}{3} \right] dx$$

$$= \int_{x=0}^1 \left[x^{5/2} - \frac{4}{3} x^3 + \frac{x^{3/2}}{3} \right] dx$$

$$= \left[\frac{x^{5/2+1}}{5/2+1} - \frac{4}{3} \frac{x^4}{4} + \frac{1}{3} \frac{x^{3/2+1}}{3/2+1} \right]_{x=0}^1$$

$$= \left[\frac{2}{7} x^{7/2} - \frac{1}{3} x^4 + \frac{2}{15} x^{5/2} \right]_{x=0}^1$$

$$= \frac{2}{7} (1-0) - \frac{1}{3} (1-0) + \frac{2}{15} (1-0) \Rightarrow \boxed{I = \frac{3}{35}}$$

② $\int_0^1 \int_0^{\sqrt{1-y^2}} x^3 y dx dy$

→ let $I = \int_{y=0}^1 \int_{x=0}^{\sqrt{1-y^2}} x^3 y dx dy$

$$= \int_{y=0}^1 \left(\frac{x^4}{4} \cdot y \right)_{x=0}^{\sqrt{1-y^2}} dy$$

$$= \frac{1}{4} \int_{y=0}^1 y \cdot ((\sqrt{1-y^2})^4 - 0) dy$$

$$= \frac{1}{4} \int_{y=0}^1 y \cdot (1-y^2)^2 dy$$

$$= \frac{1}{4} \int_{y=0}^1 y (1 + y^4 - 2y^2) dy$$

→ $(a-b)^2$

$$\begin{aligned}
 &= \frac{1}{4} \int_{y=0}^1 [y + y^5 - 2y^3] dy \\
 &= \frac{1}{4} \left[\frac{y^2}{2} + \frac{y^6}{6} - 2 \frac{y^4}{4} \right]_{y=0}^1 \\
 &= \frac{1}{4} \left[\frac{1}{2} + \frac{1}{6} - \frac{1}{2} \right] \Rightarrow \boxed{I = \frac{1}{24}}
 \end{aligned}$$

③

$$\int_0^1 \int_x^{\sqrt{x}} xy \, dy \, dx$$

$$I = \int_{x=0}^1 \int_{y=x}^{\sqrt{x}} xy \, dy \, dx$$

$$\int_{x=0}^1 x \cdot \left[\frac{y^2}{2} \right]_{y=x}^{\sqrt{x}} dx$$

$$= \int_{x=0}^1 \frac{x}{2} [x - x^2] dx$$

$$= \int_{x=0}^1 \frac{1}{2} [x^2 - x^3] dx$$

$$= \frac{1}{2} \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_{x=0}^1$$

$$= \frac{1}{2} \left[\frac{1}{3} - \frac{1}{4} \right]$$

$$= \frac{1}{2} \left[\frac{1}{12} \right]$$

$$\boxed{I = \frac{1}{24}}$$

(4)

$$\int_{-1}^1 \int_0^2 \int_{x-z}^{x+z} (x+y+z) dy dx dz$$

→

$$\text{Let } \int_{z=-1}^1 \int_{x=0}^2 \int_{y=x-z}^{x+z} (x+y+z) dy dx dz$$

$$\Rightarrow \int_{z=-1}^1 \int_{x=0}^2 \left[x \cdot y + \frac{y^2}{2} + z \cdot y \right]_{y=x-z}^{x+z} dx dz$$

$$\int_{z=-1}^1 \int_{x=0}^2 \left[x \{x+z - x+z\} + \frac{1}{2} \{ (x+z)^2 - (x-z)^2 \} + z \{ (x+z) - (x-z) \} \right] dx dz$$

$$= \int_{z=-1}^1 \int_{x=0}^2 \left[x \cdot 2z + \frac{1}{2} \{ x^2 + z^2 + 2xz - (x^2 + z^2 - 2xz) \} + z \cdot 2z \right] dx dz$$

$$= \int_{z=-1}^1 \int_{x=0}^2 \left[2xz + \frac{1}{2} (4xz) + 2z^2 \right] dx dz$$

$$= \int_{z=-1}^1 \int_{x=0}^2 \left[2xz + 2xz + 2z^2 \right] dx dz$$

$$\int_{z=-1}^1 \int_{x=0}^2 \left[4xz + 2z^2 \right] dx dz$$

$$\int_{z=-1}^1 \left[\frac{4x^2}{2} \cdot z + 2z^2 \cdot x \right]_0^2 dz$$

$$\int_{z=-1}^1 \left[2z [2^2 - 0] + 2z^2 [2 - 0] \right] dz$$

$$\int_{z=-1}^1 4z^3 dz$$

$$\int_{z=-1}^1 \left[\frac{4x \cdot z^4}{4} \right]_{z=-1}^1$$

$$= (1)^4 - (-1)^4 \Rightarrow 1 - 1 \Rightarrow \boxed{I=0}$$

(5)

$$I = \int_{-c}^c \int_{-b}^b \int_{-a}^a (x^2 + y^2 + z^2) dx dy dz$$

→

$$\text{Let } I = \int_{z=-c}^c \int_{y=-b}^b \int_{x=-a}^a (x^2 + y^2 + z^2) dx dy dz$$

$$I = \int_{z=-c}^c \int_{y=-b}^b \left[\frac{x^3}{3} + y^2 \cdot x + z^2 \cdot x \right]_{x=-a}^a dy dz$$

$$= \int_{z=-c}^c \int_{y=-b}^b \left[\frac{1}{3} [a^3 - (-a)^3] + y^2 [a - (-a)] + z^2 [a - (-a)] \right] dy dz$$

$$= \int_{z=-c}^c \int_{y=-b}^b \left[\frac{1}{3} \cdot 2a^3 + y^2 \cdot 2a + z^2 \cdot 2a \right] dy dz$$

$$= \int_{z=-c}^c \left[\frac{2a^3}{3} \cdot y + 2a \frac{y^3}{3} + 2az^2 \cdot y \right]_{y=-b}^b dz$$

$$= \int_{z=-c}^c \left[\frac{2a^3}{3} (b - (-b)) + \frac{2a}{3} (b^3 - (-b)^3) + 2az^2 (b - (-b)) \right] dz$$

$$= \int_{z=-c}^c \left[\frac{4a^3b}{3} + \frac{4ab^3}{3} + 4ab^2z^2 \right] dz$$

$$= \left[\frac{4a^3b}{3} \cdot z + \frac{4ab^3}{3} \cdot z + 4ab \cdot \frac{z^3}{3} \right]_{z=-c}^c$$

$$= \frac{4a^3b}{3} [c - (-c)] + \frac{4ab^3}{3} [c - (-c)] + \frac{4ab}{3} [(c)^3 - (-c)^3]$$

$$I = \frac{8a^3bc}{3} + \frac{8ab^3c}{3} + \frac{8abc^3}{3} = \frac{8abc}{3} (a^2 + b^2 + c^2)$$

$$(6) \quad I = \int_0^a \int_0^x \int_0^{x+y} e^{x+y+z} dz dy dx$$

$$\rightarrow I = \int_{x=0}^a \int_{y=0}^x \int_{z=0}^{x+y} e^{x+y} \cdot e^z dz dy dx$$

$$= \int_{x=0}^a \int_{y=0}^x \left[e^{x+y} \cdot (e^z)_{z=0}^{x+y} \right] dy dx$$

$$= \int_{x=0}^a \int_{y=0}^x \left[e^{x+y} (e^{x+y} - e^0) \right] dy dx$$

$$= \int_{x=0}^a \int_{y=0}^x \left[e^{2x+2y} - e^{x+y} \right] dy dx$$

$$= \int_{x=0}^a \int_{y=0}^x \left[e^{2x} \cdot e^{2y} - e^x \cdot e^y \right] dy dx$$

$$\int_{x=0}^a \left[e^{2x} \cdot \frac{e^{2y}}{2} - e^x \cdot e^y \right]_{y=0}^x dx$$

$$\int_{x=0}^a \left[\frac{1}{2} e^{2x} (e^{2x} - e^0) - e^x (e^x - e^0) \right] dx$$

$$\int_{x=0}^a \left[\frac{e^{4x}}{2} - \frac{e^{2x}}{2} - e^{2x} + e^x \right] dx$$

$$\int_{x=0}^a \left[\frac{e^{4x}}{2} - \frac{3e^{2x}}{2} + e^x \right] dx$$

$$= \left[\frac{1}{2} \cdot \frac{e^{4x}}{4} - \frac{3}{2} \frac{e^{2x}}{2} + e^x \right]_0^a$$

$$= \left[\frac{1}{8} [e^{4a} - e^0] - \frac{3}{4} [e^{2a} - e^0] + [e^a - e^0] \right]$$

$$I = \frac{e^{4a}}{8} - \frac{1}{8} - \frac{3e^{2a}}{4} + \frac{3}{4} + e^a - 1$$

$$I = \frac{e^{4a}}{8} - \frac{3e^{2a}}{4} + e^a - \frac{3}{8}$$

7

$$I = \int_0^4 \int_0^{2\sqrt{z}} \int_0^{\sqrt{4z-x^2}} dz dy dx$$

→

Rewriting given integral of w.r.t proper limits

$$I = \int_0^4 \int_0^{2\sqrt{z}} \int_0^{\sqrt{4z-x^2}} dy dx dz$$

$$= \int_{z=0}^4 \int_{x=0}^{2\sqrt{z}} [y]_0^{\sqrt{4z-x^2}} dx dz$$

$$= \int_{z=0}^4 \int_{x=0}^{2\sqrt{z}} \sqrt{4z-x^2} dx dz$$

$$\text{let } 4z = a^2 \Rightarrow a = \sqrt{4z} \Rightarrow a = 2\sqrt{z}$$

$$I = \int_{z=0}^a \int_{x=0}^a \sqrt{a^2 - x^2} dx dz$$

$$\text{Formula: } \int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right)$$

$$= \int_{z=0}^4 \left[\frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) \right]_{x=0}^a dz$$

$$= \int_{z=0}^4 \frac{a^2}{2} [\sin^{-1}(1) - \sin^{-1}(0)] dz$$

$$= \int_{z=0}^4 \frac{a^2}{2} \left(\frac{\pi}{2} - 0 \right) dz$$

$$= \frac{\pi}{4} \int_{z=0}^4 4z dz \quad + z = a^2$$

$$= \pi \left[\frac{z^2}{2} \right]_{z=0}^4$$

$$= \frac{\pi}{2} [4^2 - 0] \Rightarrow \frac{\pi}{2} \cdot 16 \Rightarrow \boxed{I = 8\pi}$$

$$(8) \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{dz dy dx}{\sqrt{a^2-x^2-y^2-z^2}}$$

$$\rightarrow \int_{x=0}^a \int_{y=0}^{\sqrt{a^2-x^2}} \int_{z=0}^{\sqrt{a^2-x^2-y^2}} \frac{1}{\sqrt{(\sqrt{a^2-x^2-y^2})^2 - z^2}} dz dy dx$$

$$\text{Let } \sqrt{a^2-x^2-y^2} = K$$

$$= \int_{x=0}^a \int_{y=0}^{\sqrt{a^2-x^2}} \int_0^K \frac{1}{\sqrt{K^2 - z^2}} dz dy dx$$

$$= \int_{x=0}^a \int_{y=0}^{\sqrt{a^2-x^2}} \left\{ \sin^{-1}\left(\frac{z}{K}\right) \right\}_{z=0}^K dy dx$$

$$= \int_{x=0}^a \int_{y=0}^{\sqrt{a^2-x^2}} \{ \sin^{-1}(1) - \sin^{-1}(0) \} dy dx$$

$$= \frac{\pi}{2} \int_{x=0}^a \int_{y=0}^{\sqrt{a^2-x^2}} 1 dy dx$$

$$\begin{aligned}
 I &= \frac{\pi}{2} \int_{x=0}^a [y]_{y=0}^{\sqrt{a^2-x^2}} dx \\
 &= \frac{\pi}{2} \int_{x=0}^a \sqrt{a^2-x^2} dx \\
 &= \frac{\pi}{2} \left[\frac{x}{2} \sqrt{a^2-x^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) \right]_{x=0}^a \\
 &= \frac{\pi}{2} \cdot \frac{a^2}{2} \left\{ \sin^{-1}(1) - \sin^{-1}(0) \right\} \\
 &= \frac{\pi a^2}{4} \cdot \frac{\pi}{2} \Rightarrow \boxed{I = \frac{\pi^2 a^2}{8}}
 \end{aligned}$$

⑨ $\int_{-3}^3 \int_0^1 \int_1^2 (x+y+z) dx dy dz = ?$

⑩ $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz dz dy dx$

⑨ Let $I = \int_{z=-3}^3 \int_{y=0}^1 \int_{x=1}^2 (x+y+z) dx dy dz$

$$\int_{z=-3}^3 \int_{y=0}^1 \left[\frac{x^2}{2} + yx + zx \right]_{x=1}^2 dy dz$$

$$\int_{z=-3}^3 \int_{y=0}^1 \left[\frac{(4-1)}{2} + y(2-1) + z(2-1) \right] dy dz$$

$$\int_{z=-3}^3 \int_{y=0}^1 \left[\frac{3}{2} + y + z \right] dy dz$$

$$\int_{z=-3}^3 \left[\frac{3}{2}y + \frac{y^2}{2} + zy \right]_{y=0}^1 dz$$

$$\int_{z=-3}^3 \left[\frac{3}{2}(1-0) + \frac{(1^2-0)}{2} + z(1-0) \right] dz$$

$$\int_{z=-3}^3 \left(\frac{3}{2} + \frac{1}{2} + z \right) dz$$

$$I = \left[\frac{3z}{2} + \frac{1z}{2} + \frac{z^2}{2} \right]_{z=-3}^3$$

$$I = \frac{3}{2} [3 - (-3)] + \frac{1}{2} [3 - (-3)] + \frac{1}{2} [3^2 - (-3)^2]$$

$$I = \frac{3 \times 6}{2} + \frac{1 \times 6}{2} + 0 \Rightarrow \boxed{I = 12}$$

(10) Let $I = \int_{x=0}^1 \int_{y=0}^{\sqrt{1-x^2}} \int_{z=0}^{\sqrt{1-x^2-y^2}} xyz \, dz \, dy \, dx$

$$\int_{x=0}^1 \int_{y=0}^{\sqrt{1-x^2}} x \cdot y \cdot \left(\frac{z^2}{2} \right)_{z=0}^{\sqrt{1-x^2-y^2}} dy \, dx$$

$$\int_{x=0}^1 \int_{y=0}^{\sqrt{1-x^2}} \left[x \cdot y \cdot \frac{1-x^2-y^2}{2} \right] dy \, dx$$

$$= \frac{1}{2} \int_{x=0}^1 \int_{y=0}^{\sqrt{1-x^2}} (xy - x^3y - xy^3) dy \, dx$$

$$= \frac{1}{2} \int_{x=0}^1 \int_{y=0}^{\sqrt{1-x^2}} \left[x \cdot \frac{y^2}{2} - x^3 \cdot \frac{y^2}{2} - x \frac{y^4}{4} \right]_{y=0}^{\sqrt{1-x^2}} dx$$

$$= \frac{1}{2} \int_{x=0}^1 \left[\frac{x}{2} [1-x^2] - \frac{x^3}{2} [1-x^2] - \frac{x}{4} (1-x^2)^2 \right] dx$$

$$= \frac{1}{2} \int_{x=0}^1 \left[\frac{x}{2} - \frac{x^3}{2} - \frac{x^3}{2} + \frac{x^5}{2} - \frac{x}{4} (1+x^4-2x^2) \right] dx$$

$$= \frac{1}{2} \int_{x=0}^1 \left[\frac{x}{2} - \frac{x^3}{2} - \frac{x^3}{2} + \frac{x^5}{2} - \frac{x}{4} - \frac{x^5}{4} + \frac{x^3}{2} \right] dx$$

$$= \frac{1}{2} \int_{x=0}^1 \left[\frac{x^2}{2} - \frac{x^4}{8} + \frac{x^6}{12} - \frac{x^2}{8} - \frac{x^6}{24} \right]_{x=0}^1$$

$$= \frac{1}{2} \left[\frac{1}{4} - \frac{1}{8} + \frac{1}{12} - \frac{1}{8} - \frac{1}{24} \right]$$

$$= \frac{1}{2} \left(-\frac{1}{24} \right)$$

$$I = \frac{1}{48}$$

Evaluation of double integral by changing the order of integration:

Evaluate the following integrals by changing the order of integration:

①

$$\int_0^1 \int_x^{\sqrt{x}} xy \, dy \, dx$$

$$\text{Let } I = \int_{x=0}^1 \int_{y=x}^{\sqrt{x}} xy \, dy \, dx$$

To identify the region of integration bounded by the curves $y=x$, $y=\sqrt{x}$ between the lines $x=0$, $x=1$

To find point of intersection:

consider, $y=x$

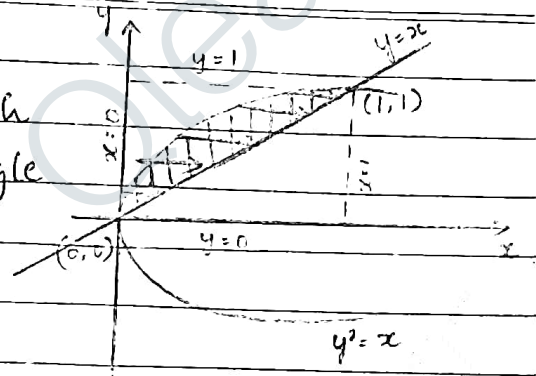
put $x=0$; $y=0$

put $x=1$; $y=1$

$\therefore$ points of intersection are $(0,0)$ and $(1,1)$

↑↑↑ limits for y
 $\Rightarrow$ limits for x

The curve $y=x$ represents a straight line passing through the origin making an angle 45° with x-axis. Also, the curve $y=\sqrt{x}$ or $y^2=x$ represents a parabola symmetrical about x-axis.



By changing the order of integration we get,

$$I = \int_{y=0}^1 \int_{x=y^2}^y xy \, dx \, dy$$

$$= \int_{y=0}^1 \left(\frac{x^2}{2} \cdot y \right)_{x=y^2}^y dy$$

$$= \frac{1}{2} \int_{y=0}^1 (y^2 - y^4) y \, dy$$

$$= \frac{1}{2} \int_{y=0}^1 (y^3 - y^5) \, dy$$

$$= \frac{1}{2} \left[\frac{y^4}{4} - \frac{y^6}{6} \right]_0^1$$

$$= \frac{1}{2} \left[\frac{1}{4} - \frac{1}{6} \right]$$

$$= \frac{1}{2} \times \frac{1}{12} \Rightarrow \boxed{I = \frac{1}{24}}$$

② $\int_0^1 \int_x^1 \frac{x}{\sqrt{x^2+y^2}} \, dy \, dx$

let $I = \int_{x=0}^1 \int_{y=x}^1 \frac{x}{\sqrt{x^2+y^2}} \, dy \, dx$

To find point of intersection:

consider $y=x$

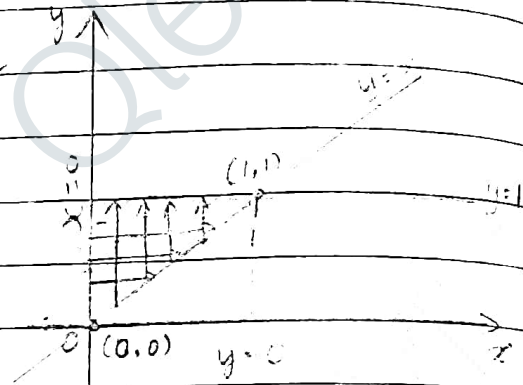
put $x=0$; $y=0$

put $x=1$; $y=1$

$\therefore$ points of intersection are $(0,0)$ and $(1,1)$

By changing the order of integration,

$$I = \int_{y=0}^1 \int_{x=0}^y \frac{x}{\sqrt{x^2+y^2}} dx dy \quad \text{--- (1)}$$



$$\text{let } x^2 + y^2 = t$$

diff w.r.t

$$2x dx = dt$$

$$x dx = dt/2$$

$$\therefore \int \frac{x}{\sqrt{x^2+y^2}} = \int \frac{dt/2}{\sqrt{t}}$$

$$= \frac{1}{2} \int t^{-1/2} dt = \frac{1}{2} \frac{t^{-1/2+1}}{-1/2+1}$$

$$= \frac{1}{2} \frac{\sqrt{t}}{1/2} \Rightarrow \sqrt{t} \Rightarrow \sqrt{x^2+y^2}$$

Sub in (1),

$$I = \int_{y=0}^1 (\sqrt{x^2+y^2})_{x=0}^y dy$$

$$I = \int_{y=0}^1 [\sqrt{y^2} - \sqrt{0+y^2}] dy$$

$$I = \int_{y=0}^1 (\sqrt{2} y - y) dy$$

$$= (\sqrt{2}-1) \int_{y=0}^1 y dy$$

$$= (\sqrt{2}-1) \left(\frac{y^2}{2} \right)_{y=0}^1$$

$$\therefore (\sqrt{2}-1) \frac{1}{2} \Rightarrow \boxed{I = \frac{\sqrt{2}-1}{2}}$$

$$\textcircled{3} \int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} xy \, dy \, dx$$

$$\text{Let } I = \int_{x=0}^{4a} \int_{y=x^2/4a}^{2\sqrt{ax}} xy \, dy \, dx.$$

$$\text{given: } x: 0 \text{ to } 4a$$

$$y: \frac{x^2}{4a} \text{ to } 2\sqrt{ax}$$

$y = \frac{x^2}{4a} \Rightarrow x^2 = 4ay$ represents a parabola symmetric about y-axis

Also, $y = 2\sqrt{ax} \Rightarrow y^2 = 4ax$ represents a parabola symmetrical about x-axis

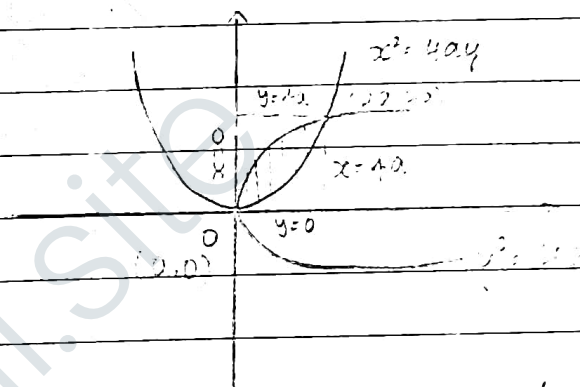
To find pt of intersection:

$$\text{consider } y = \frac{x^2}{4a}$$

$$\text{put } x=0; y=0$$

$$\text{put } x=4a; y = \frac{16a^2}{4a} = 4a$$

$\therefore$ points of intersection are $(0,0)$ and $(4a,4a)$



By changing the order of int, we get

$$I = \int_{y=0}^{4a} \int_{x=\frac{y^2}{4a}}^{2\sqrt{ay}} xy \, dx \, dy$$

$$\int_{y=0}^{4a} y \left(\frac{x^2}{2} \right)_{x=\frac{y^2}{4a}}^{2\sqrt{ay}} dy$$

$$\int_{y=0}^{4a} \frac{y}{2} \left(4ay - \frac{y^4}{16a^2} \right) dy$$

$$\int_{y=0}^{4a} \left(2ay^2 - \frac{y^5}{32a^2} \right) dy$$

$$= \left[\frac{2ay^3}{3} - \frac{1}{32a^2} \frac{y^6}{6} \right]_{y=0}^{4a}$$

$$= \frac{2a}{3} (64a^3 - 0) - \frac{1}{192a^2} (4096a^6 - 0)$$

$$= \frac{128a^4}{3} - \frac{4096a^4}{192}$$

$$= \left(\frac{128}{3} - \frac{4096}{192} \right) a^4 \Rightarrow \boxed{I = \frac{64a^4}{3}}$$

(4)

$$\int_0^a \int_y^a \frac{x}{x^2+y^2} dx dy$$

$$\text{Let } I = \int_{y=0}^a \int_{x=y}^a \frac{x}{x^2+y^2} dx dy$$

given, $y : 0 \text{ to } a$

$x : y \text{ to } a$

$x=y$ is a straight line passing through the origin making an angle 45° to x -axis

Also, $x=a$ is a straight line parallel to y -axis

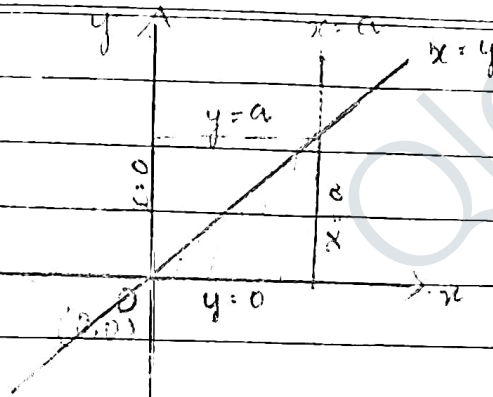
To find pts of intersection:

Consider $x=y$

put $y=0$; $x=0$

put $y=a$; $x=a$

points are $(0,0)$ and (a,a)



By changing order of integration we get,

$$I = \int_{x=0}^a \int_{y=0}^x \frac{x}{x^2+y^2} dy dx$$

$$= \int_{x=0}^a x \left\{ \int_{y=0}^x \frac{1}{y^2+x^2} dy \right\} dx$$

$$= \int_{x=0}^a x \left\{ \frac{1}{x} \cdot \tan^{-1}\left(\frac{y}{x}\right) \right\}_{y=0}^x dx \quad \left(\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) \right)$$

$$= \int_{x=0}^a (\tan^{-1}(1) - \tan^{-1}(0)) dx$$

$$= \int_{x=0}^a \left(\frac{\pi}{4} - 0 \right) dx$$

$$= \frac{\pi}{4} (x)_0^a \Rightarrow \boxed{I = \frac{\pi a}{4}}$$

$$(5) \int_0^b \int_0^{\frac{a}{b}\sqrt{b^2-y^2}} xy dx dy$$

$$\text{Let } I = \int_{y=0}^b \int_{x=0}^{\frac{a}{b}\sqrt{b^2-y^2}} xy dx dy$$

given, $y = 0$ to b

$x = 0$ to $\frac{a}{b}\sqrt{b^2-y^2}$

consider, $x = \frac{a}{b}\sqrt{b^2-y^2}$

$$= x^2 = \frac{a^2}{b^2} (b^2 - y^2)$$

$$\frac{x^2}{a^2} = \frac{1}{b^2} (b^2 - y^2)$$

$$\Rightarrow \frac{x^2}{a^2} = 1 - \frac{y^2}{b^2}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ represents ellipse}$$

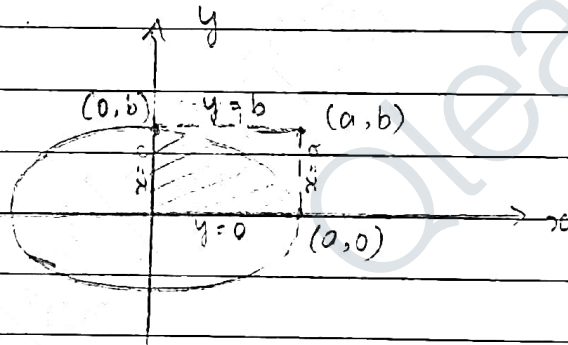
To find pts of intersection:

$$\text{Consider } x = \frac{a}{b} \sqrt{b^2 - y^2}$$

$$\text{put } y = 0 ; x = \frac{a}{b} \sqrt{b^2} = \frac{a}{b} \cdot b \Rightarrow a$$

$$\text{put } y = b ; x = \frac{a}{b} \sqrt{b^2 - b^2} \Rightarrow 0$$

pts of intersection are $(a, 0)$ and $(0, b)$.



By changing order of integration,

$$I = \int_{x=0}^a \int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} xy \, dy \, dx$$

$$= \int_{x=0}^a x \cdot \left(\frac{y^2}{2} \right)_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$= \int_{x=0}^a \frac{x}{2} \left\{ \frac{b^2}{a^2} (a^2 - x^2) \right\} dx$$

$$= \int_{x=0}^a \frac{x}{2} \left(b^2 - \frac{b^2 x^2}{a^2} \right) dx$$

$$= \int_{x=0}^a \left(\frac{b^2 x}{2} - \frac{b^2 x^3}{2a^2} \right) dx$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$\frac{y^2}{b^2} = \frac{a^2 - x^2}{a^2}$$

$$y^2 = b^2 \left(\frac{a^2 - x^2}{a^2} \right)$$

$$y = \frac{b}{a} \sqrt{a^2 - x^2}$$

$$= \left[\frac{b^2}{2} \left(\frac{x^2}{2} \right)_0^a + \frac{b^2}{2a} \left(\frac{x^4}{4} \right)_0^a \right]$$

$$= \left[\frac{b^2}{4} [a^2 - 0] - \frac{b^2}{8a^2} (a^4 - 0) \right]$$

$$= \frac{a^2 b^2}{4} - \frac{a^2 b^2}{8} \Rightarrow \boxed{I = \frac{a^2 b^2}{8}}$$

⑥ $\int_0^1 \int_{\sqrt{y}}^1 dy dx$

given $\int_{y=0}^1 \int_{x=\sqrt{y}}^1 1 dx dy$

given: $y = 0$ to 1 ; $x = \sqrt{y}$ to 1

Consider $x^2 = y$ is a parabola and $x = 1$ is a straight line parallel to y -axis

pts of intersection:

$$x^2 = y$$

$$x = 0; y = 0$$

$$x = 1; y = 1$$

pts of intersection $(0,0)$ $(1,1)$

By changing order of int,

$$I = \int_{x=0}^1 \int_{y=0}^{x^2} 1 dy dx$$

$$= \int_{x=0}^1 [y]_{y=0}^{x^2} dx$$

$$= \int_{x=0}^1 x^2 dx$$

$$= \left[\frac{x^3}{3} \right]_{x=0}^1$$

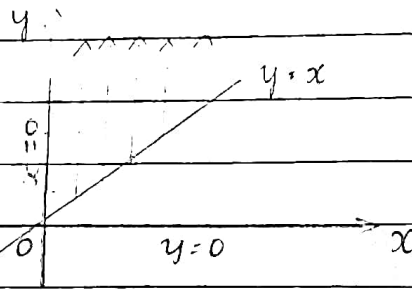
$$\Rightarrow \boxed{I = \frac{1}{3}}$$

⑦

$$\int_0^{\infty} \int_x^{\infty} \frac{e^{-y}}{y} dy dx$$

$$\text{Let } I = \int_{x=0}^{\infty} \int_{y=x}^{\infty} \frac{e^{-y}}{y} dy dx$$

$y=x$ is a straight line passing through origin making an angle 45° with x -axis



By changing order of integration.

$$I = \int_{y=0}^{\infty} \int_{x=0}^y \frac{e^{-y}}{y} dx dy$$

$$= \int_{y=0}^{\infty} \left[\frac{e^{-y}}{y} [x]_{x=0}^y \right] dy$$

$$= \int_{y=0}^{\infty} \frac{e^{-y}}{y} \cdot (y-0) dy$$

$$= \int_{y=0}^{\infty} e^{-y} dy \Rightarrow \int_{y=0}^{\infty} \left(\frac{e^{-y}}{-1} \right)_{y=0}^{\infty}$$

$$= \int_{y=0}^{\infty} -1 (e^{-\infty} - e^0)$$

$$= -1(0-1) \Rightarrow \boxed{I=1}$$

⑧

$$\int_0^1 \int_{x^2}^{2-x} xy dy dx$$

→ given $x : 0 \text{ to } 1$

$y : x^2 \text{ to } 2-x$

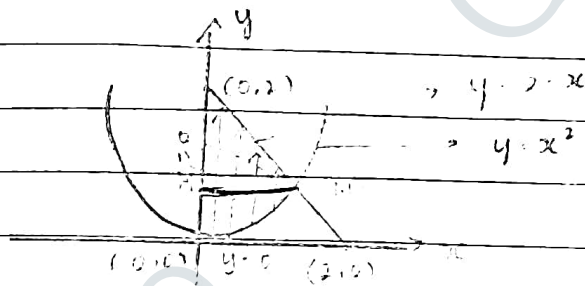
$y=x^2$ represents a parabola symmetrical about y -axis. Also, $y=2-x$ (or) $x+y=2$ (or) $\frac{x}{2} + \frac{y}{2} = 1$ represents a straight line passing through the points $(2,0)$ and $(0,2)$

10 find pts of intersection:

consider $y = x^2$

put $x=0$; $y=0 \Rightarrow (0,0) (1,1)$

$x=1$; $y=1$



By changing the order of integration,

$$I = \int_{y=0}^1 \int_{x=0}^{\sqrt{y}} xy \, dx \, dy + \int_{y=0}^2 \int_{x=0}^{2-y} xy \, dx \, dy$$

$$= \int_{y=0}^1 y \left[\frac{x^2}{2} \right]_{x=0}^{\sqrt{y}} dy + \int_{y=0}^2 y \left[\frac{x^2}{2} \right]_{x=0}^{2-y} dy$$

$$= \int_{y=0}^1 \frac{y}{2} (y) dy + \int_{y=1}^2 \frac{y}{2} \{ (2-y)^2 \} dy$$

$$= \int_{y=0}^1 \frac{y^2}{2} dy + \int_{y=1}^2 \frac{y}{2} [4 + y^2 - 4y] dy$$

$$= \int_{y=0}^1 \frac{1}{2} \frac{y^3}{3} + \int_{y=1}^2 \left(2y + \frac{y^3}{2} - 2y^2 \right) dy$$

$$= \frac{1}{2} \left[\frac{1}{3} \right] + \int_{y=1}^2 \left[2 \frac{y^2}{2} + \frac{1}{2} \frac{y^4}{4} - 2 \frac{y^3}{3} \right]_{y=1}^2$$

$$= \frac{1}{6} + \left[y^2 + \frac{1}{8} y^4 - \frac{2}{3} y^3 \right]_{y=1}^2$$

$$+ \left[(4-1) + \frac{1}{8} (16-1) - \frac{2}{3} (8-1) \right]$$

$$+ 3 + \frac{15}{8} - \frac{14}{3} \Rightarrow \frac{5}{24}$$

$$I = \frac{1}{6} + \frac{5}{24} \Rightarrow \boxed{I = \frac{3}{8}}$$

Evaluation of double integral by changing into polar form OR Change of variables

★ Use polar form of substitution $x = r \cos \theta$, $y = r \sin \theta$ so that $x^2 + y^2 = r^2$; $\frac{y}{x} = \tan \theta$

★ $dx \cdot dy = r \cdot dr \cdot d\theta$ (by using $dx \cdot dy = J \cdot dr \cdot d\theta$, where $J = \frac{\partial(x, y)}{\partial(r, \theta)} = r$)

★ change the limits of integration to (r, θ) and evaluate

Evaluate following integrals by changing into polar coordinates

① $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy$

→

In polar,

$$x = r \cos \theta, \quad y = r \sin \theta$$

$$\therefore x^2 + y^2 = r^2 \text{ and } dx dy = r dr d\theta$$

Since x, y varies from 0 to ∞ , r also varies from 0 to ∞ . Also in the first quadrant ' θ ' varies from 0 to $\pi/2$.

Converting to polar coordinates, we get

$$I = \int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} e^{-r^2} r dr d\theta \quad \text{--- (1)}$$

$$\text{put } r^2 = t$$

$$2r dr = dt \Rightarrow r dr = dt/2$$

$$\therefore \int e^{-r^2} r dr = \int e^{-t} \cdot \frac{dt}{2} = \frac{1}{2} \cdot \frac{e^{-t}}{-1} = -\frac{1}{2} e^{-r^2}$$

sub in (1),

$$I = \int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} \frac{-1}{2} (e^{-r^2}) \cdot r \, dr \, d\theta$$

$$= -\frac{1}{2} \int_{\theta=0}^{\pi/2} (e^{-\infty} - e^0) \, d\theta$$

$$= -\frac{1}{2} \int_{\theta=0}^{\pi/2} (0 - 1) \, d\theta$$

$$= \frac{1}{2} \int_{\theta=0}^{\pi/2} 1 \cdot d\theta$$

$$= \frac{1}{2} (\theta) \Big|_0^{\pi/2} = \frac{1}{2} \left(\frac{\pi}{2} - 0 \right) \Rightarrow \boxed{I = \frac{\pi}{4}}$$

$$\textcircled{2} \int_{-a}^a \int_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2} \, dy \, dx$$

→ given $x: -a \text{ to } a$

$y: 0 \text{ to } \sqrt{a^2-x^2}$

$$y = \sqrt{a^2-x^2} \Rightarrow y^2 = a^2-x^2 \Rightarrow x^2+y^2 = a^2 \text{ (circle)}$$

WKT, $x = r \cos \theta$, $y = r \sin \theta$

$$\Rightarrow x^2+y^2 = r^2$$

$$\Rightarrow a^2 = r^2 \Rightarrow \boxed{r=a}$$

$\therefore r: 0 \text{ to } a$

$\theta: 0 \text{ to } \pi$

$\therefore$ given integral is,

$$I = \int_{\theta=0}^{\pi} \int_{r=0}^a \sqrt{r^2} \cdot r \, dr \, d\theta$$

$$= \int_{\theta=0}^{\pi} \int_{r=0}^a r^2 \, dr \, d\theta$$

$$= \int_{\theta=0}^{\pi} \left[\frac{r^3}{3} \right]_{r=0}^a d\theta$$

$$= \int_{\theta=0}^{\pi} \frac{1}{3} (a^3 - 0) \, d\theta$$

$$= \frac{a^3}{3} \int_0^{\pi} 1 \cdot d\theta$$

$$= \frac{a^3}{3} (\theta)_{\theta=0}^{\pi} \Rightarrow \boxed{I = \frac{\pi a^3}{3}}$$

③ $\int_0^a \int_0^{\sqrt{a^2-y^2}} y \sqrt{x^2+y^2} \, dx \, dy$

given, $x: 0 \text{ to } \sqrt{a^2-y^2}$

$y: 0 \text{ to } a$

Consider, $x = \sqrt{a^2-y^2} \Rightarrow x^2 = a^2-y^2 \Rightarrow x^2+y^2 = a^2$ (circle)

$\therefore x: 0 \text{ to } a$

$\theta: 0 \text{ to } \pi/2$

given integral is,

$$\int_{\theta=0}^{\pi/2} \int_{x=0}^a r \sin \theta \sqrt{r^2} \cdot r \, dr \, d\theta$$

$$= \int_{\theta=0}^{\pi/2} \int_{x=0}^a r^3 \sin \theta \, dr \, d\theta$$

$$= \int_{\theta=0}^{\pi/2} \sin \theta \left(\frac{r^4}{4} \right)_{x=0}^a d\theta$$

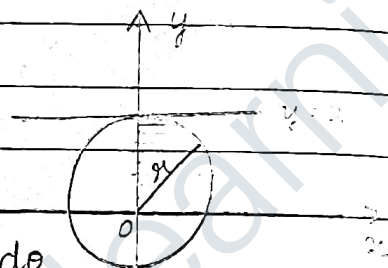
$$= \int_{\theta=0}^{\pi/2} \frac{\sin \theta}{4} (a^4) d\theta$$

$$= \frac{a^4}{4} \int_{\theta=0}^{\pi/2} \sin \theta d\theta$$

$$= \frac{a^4}{4} (-\cos \theta)_{\theta=0}^{\pi/2}$$

$$= -\frac{a^4}{4} \left[\cos \frac{\pi}{2} - \cos(0) \right]$$

$$= -\frac{a^4}{4} (0-1) \Rightarrow \boxed{I = \frac{a^4}{4}}$$



$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$\Rightarrow x^2 + y^2 = r^2$$

$$\text{also } dx dy = r dr d\theta$$

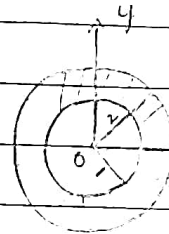
Region b/w 2 circles

④ Evaluate $\iint_R \frac{x^2 y^2}{x^2 + y^2} dx dy$ where 'R' is annular

region b/w circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 1$
convert to polar coordinates and integrate
from the fig,

clearly, $r: 1 \text{ to } 2$

$\theta: 0 \text{ to } 2\pi$



Polar conversion:

$$x = r \cos \theta; \quad y = r \sin \theta$$

$$\Rightarrow x^2 + y^2 = r^2$$

$$\text{Also } dx dy = r dr d\theta$$

$$\therefore I = \int_{\theta=0}^{2\pi} \int_{r=1}^2 \frac{r^2 \cos^2 \theta (r^2 \sin^2 \theta)}{r^2} r dr d\theta$$

$$I = \int_{\theta=0}^{2\pi} \int_{r=1}^2 r^3 \sin^2 \theta \cos^2 \theta dr d\theta$$

$$= \int_{\theta=0}^{2\pi} \sin^2 \theta \cos^2 \theta \left(\frac{r^4}{4} \right)_{r=1}^2 d\theta$$

$$= \frac{1}{4} \int_{\theta=0}^{2\pi} \sin^2 \theta \cos^2 \theta d\theta (2^4 - 1^4)$$

$$= \frac{15}{4} \int_{\theta=0}^{2\pi} \sin^2 \theta \cos^2 \theta d\theta$$

$$= \frac{15}{4} \int_{\theta=0}^{2\pi} (\sin \theta \cos \theta)^2 d\theta$$

$$= \frac{15}{4} \int_{\theta=0}^{2\pi} \left(\frac{\sin 2\theta}{2} \right)^2 d\theta$$

$$= \frac{15}{16} \int_{\theta=0}^{2\pi} \sin^2 2\theta d\theta$$

$$\text{WKT, } \sin^2 \theta = \frac{1 - \cos 2\theta}{2}$$

$$\sin^2 2\theta = \frac{1 - \cos 4\theta}{2}$$

$$\therefore I = \frac{15}{16} \int_{\theta=0}^{2\pi} \theta - \sin\left(\frac{1-\cos 4\theta}{2}\right) d\theta$$

$$= \frac{15}{32} \left[\theta - \frac{\sin 4\theta}{4} \right]_{\theta=0}^{2\pi}$$

$$= \frac{15}{32} \left[2\pi - \frac{\sin 8\pi}{4} \right] \Rightarrow \boxed{I = \frac{15\pi}{16}}$$

(5)

$$\int_0^{2a} \int_0^{\sqrt{2ax-x^2}} x^2 dy dx$$

given: $x = 0$ to $2a$

$y = 0$ to $\sqrt{2ax-x^2}$

consider, $y^2 = 2ax - x^2$

$$\Rightarrow x^2 + y^2 = 2ax$$

$$\Rightarrow x^2 + y^2 - 2ax = 0 \text{ (circle)}$$

This represents the circle eqⁿ with centre at $(a, 0)$ and radius = a

clearly, $\theta: 0$ to $\pi/2$

consider $x^2 + y^2 - 2ax = 0$

polar conversion:

$$x = r \cos \theta; \quad y = r \sin \theta$$

$$\Rightarrow x^2 + y^2 = r^2$$

$$\text{Also } dx dy = r dr d\theta$$

$$\therefore r^2 - 2ar \cos \theta = 0$$

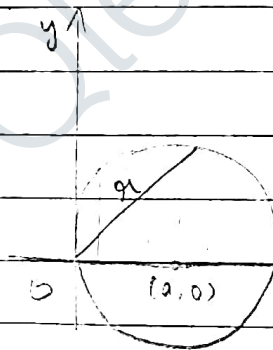
$$r(r - 2a \cos \theta) = 0$$

$$\Rightarrow \boxed{r=0}; \quad r - 2a \cos \theta$$

$$\Rightarrow \boxed{r = 2a \cos \theta}$$

$$\therefore r: 0 \text{ to } 2a \cos \theta$$

$$\therefore I = \int_{\theta=0}^{\pi/2} \int_{r=0}^{2a \cos \theta} r^2 \cancel{dx dy} r dr d\theta$$



$$= \int_{\theta=0}^{\pi/2} \int_{r=0}^{2a \cos \theta} r^3 \cos^2 \theta \, dr \, d\theta$$

$$= \int_{\theta=0}^{\pi/2} \cos^2 \theta \left[\frac{r^4}{4} \right]_{r=0}^{2a \cos \theta} d\theta$$

$$= \int_{\theta=0}^{\pi/2} \frac{\cos^2 \theta}{4} [16a^4 \cos^4 \theta] d\theta$$

$$= \int_{\theta=0}^{\pi/2} (4a^4 \cos^6 \theta) d\theta$$

$$= 4a^4 \int_{\theta=0}^{\pi/2} \cos^6 \theta d\theta$$

$$= 4a^4 \left[\frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \right] \Rightarrow \boxed{I = \frac{5\pi a^4}{8}}$$

Note: Reduction formula:

$$(i) \int_0^{\pi/2} \sin^n \theta d\theta = \int_0^{\pi/2} \cos^n \theta d\theta = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \dots \frac{1}{2} \cdot \frac{\pi}{2}$$

if n is even.

$$(ii) \int_0^{\pi/2} \sin^n \theta d\theta = \int_0^{\pi/2} \cos^n \theta d\theta = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \dots \frac{2}{3} \text{ if } n \text{ is odd}$$

$$(6) \int_0^1 \int_0^{\sqrt{1-y^2}} (x^2 + y^2) dx dy$$

given: $x = 0 \text{ to } \sqrt{1-y^2}$

$y = 0 \text{ to } 1$

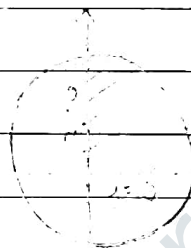
Consider, $x = \sqrt{1-y^2}$

$$x^2 = 1 - y^2 \Rightarrow x^2 + y^2 = 1 \text{ (circle)}$$

Polar conversion,

$$x = r \sin \theta; y = r \cos \theta; x^2 + y^2 = r^2$$

Also, $dx dy = r dr d\theta$



$$\therefore \theta = 0 \text{ to } \pi/2$$

$$r = 0 \text{ to } 1$$

By changing to polar form

$$I = \int_{\theta=0}^{\pi/2} \int_{r=0}^1 r \, r \, dr \, d\theta$$

$$= \int_{\theta=0}^{\pi/2} \int_{r=0}^1 r^3 \, dr \, d\theta$$

$$= \int_{\theta=0}^{\pi/2} \left[\frac{r^4}{4} \right]_{r=0}^1 d\theta$$

$$= \int_{\theta=0}^{\pi/2} \frac{1}{4} d\theta$$

$$= \frac{1}{4} \int_{\theta=0}^{\pi/2} 1 \cdot d\theta \Rightarrow \frac{1}{4} [\theta]_{\theta=0}^{\pi/2} = \frac{1}{4} \left[\frac{\pi}{2} \right] \Rightarrow \boxed{I = \frac{\pi}{8}}$$

Applications to find area and volume by double integrals

Formulae:

(i) $\iint_R dx \, dy = \text{area of the region 'R' in cartesian form}$

(ii) $\iint_R r \, dr \, d\theta = \text{area of the region capital 'R' in the polar form}$

(iii) $\iiint_V z \, dy \, dx = \text{volume of the solid in the cartesian form.}$

① Find the area of the ellipse by double integration.

→ Area $A = \iint_R dx dy$

we have $A = 4A_1$, where A_1 is the area in the 1st quadrant

$$\therefore A = 4A_1 = 4 \int_{x=0}^a \int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} 1 dy dx$$

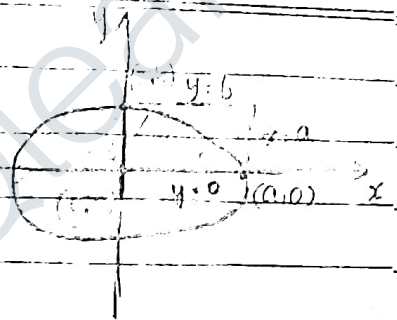
$$A = 4 \int_{x=0}^a [y]_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$= 4 \int_{x=0}^a \frac{b}{a} \sqrt{a^2-x^2} dx$$

$$= \frac{4b}{a} \left[\frac{x}{2} \sqrt{a^2-x^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) \right]_{x=0}^a$$

$$= \frac{4b}{a} \cdot \frac{a^2}{2} [\sin^{-1}(1) - \sin^{-1}(0)]$$

$$= 2ba \left[\frac{\pi}{2} - 0 \right] \Rightarrow \boxed{A = \pi ab \text{ sq. units}}$$



we have,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$\frac{y^2}{b^2} = \frac{a^2-x^2}{a^2}$$

$$y^2 = \frac{b^2}{a^2} (a^2-x^2)$$

$$y = \frac{b}{a} \sqrt{a^2-x^2}$$

- ② Find the area bounded b/w parabolas $y^2 = 4ax$ and $x^2 = 4ay$

To find pts of intersection:

from the curve $y^2 = 4ax$, we have

$$x = \frac{y^2}{4a}$$

sub this x in $x^2 = 4ay$

$$\text{we get, } \left(\frac{y^2}{4a}\right)^2 = 4ay$$

$$\frac{y^4}{16a^2} = 4ay$$

$$y^4 = 64a^3y$$

$$y^4 - 64a^3y = 0$$

$$y(y^3 - 64a^3) = 0$$

$$\Rightarrow \boxed{y=0} \quad y^3 = 64a^3 \Rightarrow \boxed{y=4a}$$

Consider,

$$x = \frac{y^2}{4a}$$

put $y=0$; $x=0$ put $y=4a$; $x=4a$ pts are $(0,0)$ $(4a,4a)$

$$\text{Area, } A = \int_{x=0}^{4a} \int_{y=\frac{x^2}{4a}}^{2\sqrt{ax}} 1 \, dy \, dx$$

$$= \int_{x=0}^{4a} (y) \frac{x^2}{4a} \, dx$$

$$= \int_{x=0}^{4a} \left[2\sqrt{ax} - \frac{x^2}{4a} \right] dx$$

$$= \left[2\sqrt{a} \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{1}{4a} \frac{x^3}{3} \right]_{x=0}^{4a}$$

$$= \left[2\sqrt{a} \frac{2}{3} x^{3/2} - \frac{1}{4a} \frac{x^3}{3} \right]_{x=0}^{4a}$$

$$= \left[2\sqrt{a} \frac{2}{3} (4a)^{3/2} - 0 - \frac{1}{12a} 64a^3 \right]$$

$$= \left[\frac{4\sqrt{a}}{3} \sqrt{64} a^{3/2} - \frac{1}{12a} 64a^3 \right]$$

$$= \frac{4}{3} a^2 \cdot 8 - \frac{64}{12} a^2 \Rightarrow a^2 \left(\frac{32}{3} - \frac{64}{12} \right) \Rightarrow \boxed{I = \frac{16}{3} a^2}$$

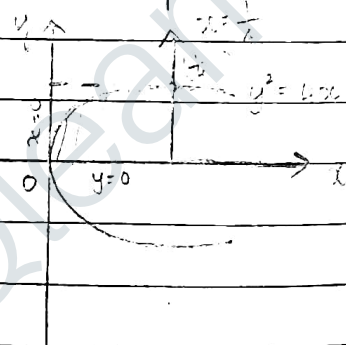
③ Evaluate $\iint_R dx \, dy$ over the region bounded by the parabola $y^2 = 4x$ and line $x = \frac{1}{4}$

→ To find pts of intersection:

consider $y^2 = 4x$

put $x = \frac{1}{4}$; $y = 1$

pts is $\left(\frac{1}{4}, 1\right)$



$$\text{Area, } A = \int_{x=0}^{\frac{1}{4}} \int_{y=0}^{2\sqrt{x}} 1 \cdot dy \, dx$$

$$\int_{x=0}^{\frac{1}{4}} [y]_{y=0}^{2\sqrt{x}} \, dx$$

$$= \int_{x=0}^{\frac{1}{4}} 2\sqrt{x} \, dx$$

$$= \int_{x=0}^{\frac{1}{4}} 2x^{\frac{1}{2}} \, dx$$

$$= \left[2 \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} \right]_{x=0}^{\frac{1}{4}}$$

$$= \frac{4}{3} [x^{\frac{3}{2}}]_{x=0}^{\frac{1}{4}}$$

$$= \frac{4}{3} \left(\frac{1}{4} \right)^{\frac{3}{2}} = \frac{4}{3} \left(\frac{1}{\sqrt{64}} \right) = \frac{4}{3} \left(\frac{1}{8} \right) \Rightarrow \boxed{I = \frac{1}{6}}$$

- (4) Find by double integration, the area lying b/w the circle $x^2 + y^2 = a^2$ and the line $x + y = a$ in the 1st quadrant.

$$\rightarrow \therefore \text{Area, } A = \iint_R dx \, dy$$

$$\int_{x=0}^a \int_{y=a-x}^{\sqrt{a^2-x^2}} 1 \cdot dy \, dx$$

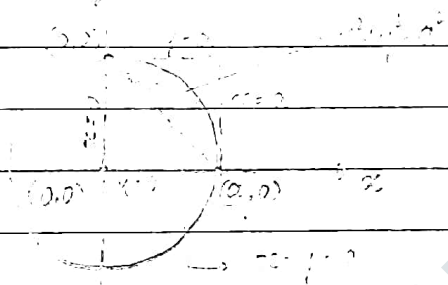
$$\int_{x=0}^a [y]_{a-x}^{\sqrt{a^2-x^2}} \, dx$$

$$\int_{x=0}^a \left[\sqrt{a^2-x^2} - (a-x) \right] \, dx$$

$$\int_{x=0}^a \left(\frac{x}{2} \sqrt{a^2-x^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) - ax + \frac{x^2}{2} \right)_{x=0}^a$$

$$\int_{x=0}^a \frac{a^2}{2} \left[\sin^{-1}(1) - \sin^{-1}(0) \right] - a^2 + \frac{1}{2} a^2$$

$$\frac{a^2}{2} \left(\frac{\pi}{2} \right) - \frac{a^2}{2} \Rightarrow \frac{\pi a^2}{4} - \frac{a^2}{2} \Rightarrow \boxed{I = \left(\frac{\pi-1}{4} \right) a^2 \text{ sq units}}$$



- (5) Find the area enclosed by following curve.
- (i) one loop of the lemniscate $r^2 = a^2 \cos 2\theta$
- (ii) upper part of the cardioid $r = a(1 + \cos \theta)$

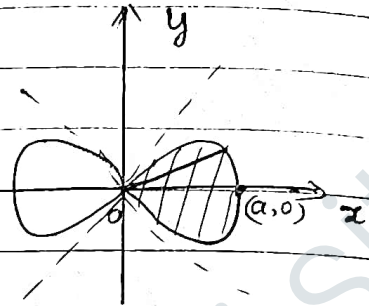
(i) Lemniscate is as shown below:
area enclosed by one loop of this curve,

$A = 2 \times$ area bounded by the curve in first quadrant

In the first quadrant,

$$\theta : 0 \text{ to } \pi/4$$

$$r : 0 \text{ to } a\sqrt{\cos 2\theta}$$



$$\therefore A = 2 \times \int_{\theta=0}^{\pi/4} \int_{r=0}^{a\sqrt{\cos 2\theta}} r \, dr \, d\theta$$

$$A = 2 \times \int_{\theta=0}^{\pi/4} \left(\frac{r^2}{2} \right)_{r=0}^{a\sqrt{\cos 2\theta}} d\theta$$

$$= 2 \times \int_{\theta=0}^{\pi/4} \frac{1}{2} (a^2 \cos 2\theta - 0) d\theta$$

$$= 2 \times \frac{a^2}{2} \int_{\theta=0}^{\pi/4} \cos 2\theta d\theta$$

$$= a^2 \left[\frac{\sin 2\theta}{2} \right]_{\theta=0}^{\pi/4}$$

$$= \frac{a^2}{2} \left[\sin\left(\frac{\pi}{2}\right) - \sin(0) \right]$$

$$\boxed{A = \frac{a^2}{2} \text{ sq units}}$$

(ii) Required area = $\iint_R r \, dr \, d\theta$

$\theta: 0 \text{ to } \pi$

$r: 0 \text{ to } a(1+\cos\theta)$

$\int_{\theta=0}^{\pi} \int_{r=0}^{a(1+\cos\theta)} r \, dr \, d\theta$

$\int_{\theta=0}^{\pi} \left(\frac{r^2}{2} \right)_{r=0}^{a(1+\cos\theta)} d\theta$

$\int_{\theta=0}^{\pi} \frac{1}{2} a^2 (1+\cos\theta)^2 d\theta$

$= \frac{1}{2} \int_{\theta=0}^{\pi} a^2 [1 + \cos^2\theta + 2\cos\theta] d\theta$

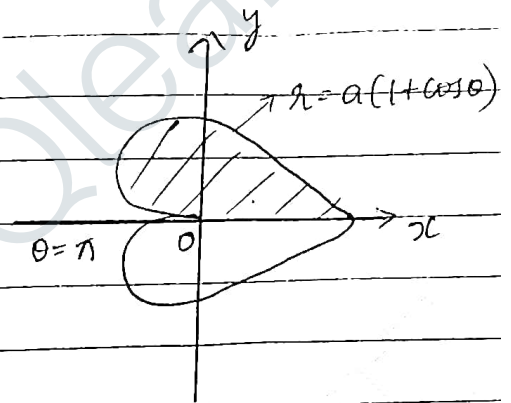
$= \frac{a^2}{2} \int_{\theta=0}^{\pi} \left[1 + \left(\frac{1+\cos 2\theta}{2} \right) + 2\cos\theta \right] d\theta$

$= \frac{a^2}{2} \left[\theta + \frac{1}{2}\theta + \frac{1}{2} \frac{\sin 2\theta}{2} + 2\sin\theta \right]_{\theta=0}^{\pi}$

$= \frac{a^2}{2} \left[\pi + \frac{\pi}{2} + \frac{1}{4} [\sin 2\pi - \sin 0] + 2[\sin \pi - \sin 0] \right]$

$= \frac{a^2}{2} \left[\frac{3\pi}{2} \right]$

$A = \frac{3a^2\pi}{4} \text{ sq units}$



- ⑥ A pyramid is bounded by 3 coordinate planes and the plane $x+2y+3z=6$ compute the volume by double integration.

→ Volume by double integration is given by $V = \iint z \, dx \, dy$

consider $x+2y+3z=6$

$\Rightarrow z = \frac{6-x-2y}{3} = 2 - \frac{x}{3} - \frac{2y}{3}$

If $z=0$, then $x+2y=6 \Rightarrow y = 3 - \frac{x}{2}$

If $z=0$ and $y=0$, then $x=6$

$\therefore x: 0 \text{ to } 6$

$y: 0 \text{ to } 3 - \frac{x}{2}$

$$\therefore V = \int_{x=0}^6 \int_{y=0}^{3-x/2} \left[2 - \frac{x}{3} - \frac{2y}{3} \right] dy dx$$

$$V = \int_{x=0}^6 \left[2y - \frac{x}{3}(y) - \frac{2}{3} \cdot \frac{y^2}{2} \right]_{y=0}^{3-x/2} dx$$

$$V = \int_{x=0}^6 \left[2\left(3 - \frac{x}{2}\right) - \frac{x}{3}\left(3 - \frac{x}{2}\right) - \frac{2}{6} \left(3 - \frac{x}{2}\right)^2 \right] dx$$

$$V = \int_{x=0}^6 \left[6 - x - x + \frac{x^2}{6} - \frac{1}{3} \left(9 + \frac{x^2}{4} - 3x \right) \right] dx$$

$$= \int_{x=0}^6 \left[6 - 2x + \frac{x^2}{6} - 3 - \frac{x^2}{12} + x \right] dx$$

$$= \int_{x=0}^6 \left[\frac{x^2}{12} - x + 3 \right] dx$$

$$= \left[\frac{1}{12} \frac{x^3}{3} - \frac{x^2}{2} + 3x \right]_{x=0}^6$$

$$= \frac{1}{36} [216] - \frac{1}{2} [36] + 3(6)$$

$$= 6 - 18 + 18 \Rightarrow \boxed{A = 6 \text{ Sq units}}$$

(7)

Find the volume of the tetrahedron by double integration bounded by the planes $x=0$, $y=0$, $z=0$, $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

→

given $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

$$\frac{z}{c} = 1 - \frac{x}{a} - \frac{y}{b} \Rightarrow z = c \left(1 - \frac{x}{a} - \frac{y}{b} \right)$$

put $z=0$; $\frac{x}{a} + \frac{y}{b} = 1 \Rightarrow y = b \left(1 - \frac{x}{a}\right)$

put $y=0, z=0$; $\frac{x}{a} = 1 \Rightarrow \boxed{x=a}$

$\therefore x : 0 \text{ to } a$

$y : 0 \text{ to } b \left(1 - \frac{x}{a}\right)$

Thus, volume, $V = \iiint z \, dy \, dx$

$$V = \int_{x=0}^a \int_{y=0}^{b(1-\frac{x}{a})} c \left(1 - \frac{x}{a} - \frac{y}{b}\right) dy \, dx$$

$$= c \int_{x=0}^a \left[y - \frac{cx}{a}(y) - \frac{1}{b} \frac{y^2}{2} \right]_{y=0}^{b(1-\frac{x}{a})} dx$$

$$= c \int_{x=0}^a \left[b \left(1 - \frac{x}{a}\right) - \frac{cx}{a} b \left(1 - \frac{x}{a}\right) - \frac{1}{2b} b^2 \left(1 - \frac{x}{a}\right)^2 \right] dx$$

$$= c \int_{x=0}^a \left[b - \frac{b^2 x}{a} - \frac{b^2 x}{a} + \frac{b^2 x^2}{a^2} - \frac{b}{2} \left(1 + \frac{x^2}{a^2} - \frac{2x}{a}\right) \right] dx$$

$$= c \int_{x=0}^a \left[\cancel{b} - \frac{2b^2 x}{a} + \frac{b^2 x^2}{a^2} - \frac{b}{2} - \frac{b^2 x^2}{2a^2} + \frac{bx}{a} \right] dx$$

$$= c \int_{x=0}^a \left[\frac{b}{2} - \frac{bx}{a} + \frac{1}{2} \frac{bx^2}{a^2} \right] dx$$

$$= c \int_{x=0}^a \left[\frac{bx}{2} - \frac{b}{a} \frac{x^2}{2} + \frac{b}{2a^2} \frac{x^3}{3} \right]_{x=0}^a$$

$$= c \left[\frac{b}{2} (a) - \frac{b}{2a} (a^2) + \frac{b}{6a^2} (a^3) \right]$$

$$= c \left[\cancel{\frac{ab}{2}} - \cancel{\frac{ab}{2}} + \frac{ab}{6} \right]$$

$$\boxed{A = \frac{abc}{6} \text{ sq units}}$$

Note: Volume of triple integral is given by,

$$V = \iiint dx \, dy \, dz$$

Beta and Gamma functions:

Def:

(i) $\beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx \quad (m, n > 0)$ — (1)

(ii) Sub $x = \sin^2 \theta$ and simplifying we get

$$\beta(m, n) = 2 \int_0^{\pi/2} \sin^{2m-1} \theta \cdot \cos^{2n-1} \theta d\theta \quad \text{--- (2)}$$

(iii) $\Gamma(n) = \int_0^{\infty} e^{-x} \cdot x^{n-1} \cdot dx \quad (n > 0)$ — (3)

(iv) Sub $x = t^2$ and simplifying we get

$$\Gamma(n) = 2 \int_0^{\infty} t^{2n-1} \cdot e^{-t^2} dt \quad \text{--- (4)}$$

Eqⁿ (1) and (2) are definitions of β function,
eqⁿ (3) and (4) are definitions of Γ function

properties:

(i) $\beta(m, n) = \beta(n, m)$

(ii) $\Gamma(n+1) = n \Gamma(n) = n!$ for a +ve integer 'n'

(iii) $\beta(m, n) = \frac{\Gamma(m) \cdot \Gamma(n)}{\Gamma(m+n)}$ is the relation b/w β and Γ function

(iv) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

(v) $\Gamma\left(\frac{1}{4}\right) \cdot \Gamma\left(\frac{3}{4}\right) = \pi\sqrt{2}$

(vi) from eqⁿ (2), sub $2m-1 = p$; $2n-1 = q$
 $\Rightarrow m = \frac{p+1}{2}$; $n = \frac{q+1}{2}$

eqⁿ (2) $\Rightarrow \beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = 2 \int_0^{\pi/2} \sin^p \theta \cdot \cos^q \theta d\theta$

Relation b/w Beta and Gamma function:

with usual notations prove that

$$\beta(m, n) = \frac{\Gamma(m) \cdot \Gamma(n)}{\Gamma(m+n)}$$

proof: By definition $\Gamma(n) = 2 \int_0^{\infty} x^{2n-1} \cdot e^{-x^2} dx$ — (x)

$$\Gamma(m) = 2 \int_0^{\infty} y^{2m-1} \cdot e^{-y^2} dy$$

$$\Gamma(m) \cdot \Gamma(n) = 2 \int_0^{\infty} y^{2m-1} \cdot e^{-y^2} dy \times 2 \int_0^{\infty} x^{2n-1} \cdot e^{-x^2} dx$$

$$\Gamma(m) \cdot \Gamma(n) = 4 \int_0^{\infty} \int_0^{\infty} x^{2n-1} \cdot y^{2m-1} \cdot e^{-(x^2+y^2)} dx dy \quad \text{--- (1)}$$

In polar, $x = r \cos \theta$, $y = r \sin \theta$
 $\Rightarrow x^2 + y^2 = r^2$

Also $dx dy = r dr d\theta$

$$\therefore \theta = 0 \text{ to } \pi/2$$

$$r = 0 \text{ to } \infty$$

sub in eqn (1)

$$\Gamma(m) \cdot \Gamma(n) = 4 \int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} (r \cos \theta)^{2n-1} \cdot (r \sin \theta)^{2m-1} \cdot e^{-r^2} \cdot r dr d\theta$$

$$\Gamma(m) \cdot \Gamma(n) = 4 \int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} r^{2n-1} \cdot (\cos \theta)^{2n-1} \cdot r^{2m-1} \cdot \sin^{2m-1} \theta \cdot e^{-r^2} \cdot r dr d\theta$$

$$\Gamma(m) \cdot \Gamma(n) = \left(2 \int_{\theta=0}^{\pi/2} \sin^{2m-1} \theta \cdot \cos^{2n-1} \theta d\theta \right) \times \left(2 \int_{r=0}^{\infty} r^{2(m+n)-1} \cdot e^{-r^2} dr \right)$$

$$\Gamma(m) \cdot \Gamma(n) = \beta(m, n) \times \Gamma(m+n)$$

$$\Rightarrow \beta(m, n) = \frac{\Gamma(m) \cdot \Gamma(n)}{\Gamma(m+n)}$$

Prove that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

proof:

By defⁿ $\Gamma(n) = 2 \int_0^{\infty} x^{2n-1} e^{-x^2} dx$

put $n = \frac{1}{2}$

$$\Gamma\left(\frac{1}{2}\right) = 2 \int_0^{\infty} x^0 e^{-x^2} dx$$

$$\text{i.e., } \Gamma\left(\frac{1}{2}\right) = 2 \int_0^{\infty} e^{-x^2} dx$$

$$\text{ii. } \Gamma\left(\frac{1}{2}\right) = 2 \int_0^{\infty} e^{-y^2} dy$$

Multiplying,

$$\Gamma\left(\frac{1}{2}\right) \cdot \Gamma\left(\frac{1}{2}\right) = 2 \int_0^{\infty} e^{-x^2} dx \times 2 \int_0^{\infty} e^{-y^2} dy$$

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 4 \int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)} dx dy \quad \text{--- (1)}$$

In polars, $x = r \cos \theta$; $y = r \sin \theta$

$$\Rightarrow x^2 + y^2 = r^2$$

$$\text{Also } dx dy = r dr d\theta$$

$$\theta: 0 \text{ to } \pi/2$$

$$r: 0 \text{ to } \infty$$

sub in eqⁿ (1),

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 4 \int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} e^{-r^2} r dr d\theta \quad \text{--- (2)}$$

put $r^2 = t$

$$\Rightarrow 2r dr = dt \Rightarrow r dr = dt/2$$

$$\therefore \int e^{-r^2} r dr = \int \frac{e^{-t}}{2} dt = \frac{1}{2} \left(\frac{e^{-t}}{-1} \right) = -\frac{1}{2} e^{-r^2}$$

sub in eqⁿ (2),

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 4 \int_{\theta=0}^{\pi/2} \left[-\frac{1}{2} (e^{-r^2}) \right]_0^{\infty} d\theta$$

$$= -2 \int_{\theta=0}^{\pi/2} (e^{-\infty} - e^0) d\theta$$

$$= 2 \int_{\theta=0}^{\pi/2} 1 \cdot d\theta$$

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 2 [\theta]_0^{\pi/2}$$

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 2 \left[\frac{\pi}{2}\right]$$

$$\Rightarrow \underline{\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}}$$

problems on Beta and Gamma functions

Note: (i) For expressions of the form $(a-x^n)$, substitute;
 $x^n = a \sin^2 \theta$

(ii) For expressions of the form $(a+x^n)$; substitute
 $x^n = a \tan^2 \theta$.

Evaluate the following:

$$\textcircled{1} \quad \text{s.t.} \int_0^{\infty} \sqrt{y} \cdot e^{-y^2} dy \times \int_0^{\infty} \frac{e^{-y^2}}{\sqrt{y}} dy = \frac{\pi}{2\sqrt{2}}$$

$$\text{Let } I_1 = \int_0^{\infty} \sqrt{y} \cdot e^{-y^2} dy = \int_0^{\infty} y^{1/2} \cdot e^{-y^2} dy \quad \text{--- (1)}$$

$$I_2 = \int_0^{\infty} \frac{e^{-y^2}}{\sqrt{y}} dy = \int_0^{\infty} y^{-1/2} e^{-y^2} dy \quad \text{--- (2)}$$

$$\text{By def}^n \quad \Gamma(n) = 2 \int_0^{\infty} x^{2n-1} \cdot e^{-x^2} dx$$

$$\Rightarrow \int_0^{\infty} x^{2n-1} \cdot e^{-x^2} dx = \frac{1}{2} \Gamma(n) \quad \text{--- (3)}$$

comparing (1) and (3),

$$2n-1 = \frac{1}{2} \Rightarrow 2n = \frac{3}{2} \Rightarrow \boxed{n = \frac{3}{4}}$$

$$\therefore \boxed{I_1 = \frac{1}{2} \Gamma\left(\frac{3}{4}\right)}$$

Comparing ② and ③,

$$2n-1 = -\frac{1}{2} \Rightarrow 2n = \frac{1}{2} \Rightarrow \boxed{n = \frac{1}{4}}$$

$$\therefore \boxed{I_2 = \frac{1}{2} \Gamma\left(\frac{1}{4}\right)}$$

$$\text{Thus } I_1 \times I_2 = \frac{1}{2} \Gamma\left(\frac{3}{4}\right) \times \frac{1}{2} \Gamma\left(\frac{1}{4}\right) \quad \left(\Gamma\left(\frac{3}{4}\right) \Gamma\left(\frac{1}{4}\right) = \frac{\pi}{\sqrt{2}}\right)$$

$$= \frac{1}{4} \pi \sqrt{2}$$

$$= \frac{\pi \sqrt{2}}{2 \sqrt{2} \sqrt{2}} \Rightarrow \frac{\pi}{2 \sqrt{2}} //$$

$$\textcircled{2} \quad \text{ST} \int_0^{\pi/2} \sqrt{\cot \theta} \, d\theta = \frac{\pi}{\sqrt{2}}$$

$$\text{Let } I = \int_0^{\pi/2} \sqrt{\cot \theta} \, d\theta$$

$$= \int_0^{\pi/2} \sqrt{\frac{\cos \theta}{\sin \theta}} \, d\theta = \int_0^{\pi/2} \sin^{-1/2} \theta \cos^{1/2} \theta \, d\theta$$

$$(\text{prop 6}) \quad \text{Using } \int_0^{\pi/2} \sin^p \theta \cos^q \theta \, d\theta = \frac{1}{2} \beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$\text{Here, } p = -1/2, \quad q = 1/2$$

$$\therefore I = \frac{1}{2} \beta\left(\frac{-1/2+1}{2}, \frac{1/2+1}{2}\right) = \frac{1}{2} \beta\left(\frac{1}{4}, \frac{3}{4}\right)$$

$$\Rightarrow \frac{1}{2} \frac{\Gamma(1/4) \Gamma(3/4)}{\Gamma(1/4 + 3/4)} \quad \left[\beta(m, n) = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)} \right]$$

$$= \frac{1}{2} \frac{(\pi \sqrt{2})}{\Gamma(1)}$$

$$= \frac{1}{2} \pi \sqrt{2} \quad \left[\begin{array}{l} \text{WKT } \Gamma(n+1) = n! \\ \Gamma(1) = 0! = 1 \end{array} \right]$$

$$= \frac{\pi \sqrt{2}}{\sqrt{2} \sqrt{2}}$$

$$\boxed{I = \frac{\pi}{\sqrt{2}}}$$

$$\text{Q.1} \int_0^{\pi/2} \frac{dx}{\sqrt{\sin x}} \times \int_0^{\pi/2} \sqrt{\sin x} dx = \pi$$

$$\text{Let } I_1 = \int_0^{\pi/2} \frac{dx}{\sqrt{\sin x}} \quad \int_0^{\pi/2} \sin^{1/2} x \cdot \cos^0 x dx \quad (1)$$

$$I_2 = \int_0^{\pi/2} \sqrt{\sin x} dx = \int_0^{\pi/2} \sin^{1/2} x \cdot \cos^0 x dx \quad (2)$$

$$\text{Wkt, } \int_0^{\pi/2} \sin^p x \cdot \cos^q x dx = \frac{1}{2} \beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$\text{from (1), } p = -1/2, q = 0$$

$$\therefore I_1 = \frac{1}{2} \beta\left(-\frac{1/2+1}{2}, \frac{0+1}{2}\right) = \frac{1}{2} \beta\left(-\frac{1}{4}, \frac{1}{2}\right)$$

$$\text{from (2), } p = 1/2, q = 0$$

$$\therefore I_2 = \frac{1}{2} \beta\left(\frac{1/2+1}{2}, \frac{0+1}{2}\right) = \frac{1}{2} \beta\left(\frac{3}{4}, \frac{1}{2}\right)$$

$$\therefore I_1 \times I_2 = \frac{1}{2} \beta\left(\frac{1}{4}, \frac{1}{2}\right) \times \frac{1}{2} \beta\left(\frac{3}{4}, \frac{1}{2}\right)$$

$$= \frac{1}{4} \left[\frac{\Gamma(1/4) \cdot \Gamma(1/2)}{\Gamma(1/4 + 1/2)} \times \frac{\Gamma(3/4) \cdot \Gamma(1/2)}{\Gamma(3/4 + 1/2)} \right]$$

$$= \frac{1}{4} \left[\frac{\Gamma(1/4) \cdot \sqrt{\pi}}{\Gamma(3/4)} \times \frac{\Gamma(3/4) \cdot \sqrt{\pi}}{\Gamma(5/4)} \right]$$

$$= \frac{\pi}{4} \left[\frac{\Gamma(1/4)}{\Gamma(5/4)} \right]$$

$$\text{But } \Gamma\left(\frac{5}{4}\right) = \Gamma\left(\frac{1}{4} + 1\right) = \frac{1}{4} \Gamma\left(\frac{1}{4}\right) \quad \left[\Gamma(n+1) = n \Gamma(n) \right]$$

$$I_1 \times I_2 = \frac{\pi}{4} \cdot \frac{\Gamma(1/4)}{\frac{1}{4} \Gamma(1/4)}$$

$$\Rightarrow \boxed{I_1 \times I_2 = \pi}$$

(4)

$$\int_0^{\pi/2} \sqrt{\tan \theta} d\theta$$

$$\text{Let } I = \int_0^{\pi/2} \sqrt{\tan \theta} d\theta$$

$$= \int_0^{\pi/2} \sqrt{\frac{\sin \theta}{\cos \theta}} d\theta = \int_0^{\pi/2} \sin^{1/2} \theta \cdot \cos^{-1/2} \theta d\theta$$

$$\text{Here } p = 1/2, \quad q = -1/2$$

$$I = \frac{1}{2} B\left(\frac{1/2+1}{2}, \frac{-1/2+1}{2}\right) = \frac{1}{2} B\left(\frac{3}{4}, \frac{1}{4}\right)$$

$$= \frac{1}{2} \frac{\Gamma(3/4) \cdot \Gamma(1/4)}{\Gamma(3/4 + 1/4)}$$

$$= \frac{1}{2} \frac{\pi \sqrt{2}}{\Gamma(1)}$$

$$= \frac{1}{2} \pi \sqrt{2} \Rightarrow \frac{\pi \sqrt{2}}{\sqrt{2} \sqrt{2}} \Rightarrow \boxed{I = \frac{\pi}{\sqrt{2}}}$$

(5)

$$\int_0^1 x^{3/2} (1-x)^{1/2} dx$$

$$\text{Let } I = \int_0^1 x^{3/2} (1-x)^{1/2} dx$$

for expressions of the form $(a-x^n)$ we substitute

$$x^n = a \sin^2 \theta$$

$$\text{here, } a=1, \quad n=1$$

$$\therefore x = \sin^2 \theta$$

$$dx = 2 \sin \theta \cos \theta d\theta$$

$$\therefore x^{3/2} = (\sin^2 \theta)^{3/2} = \sin^3 \theta$$

$$\text{Also, } (1-x)^{1/2} = (1-\sin^2 \theta)^{1/2} = (\cos^2 \theta)^{1/2} = \cos \theta$$

$$\text{consider } x = \sin^2 \theta$$

$$\text{put } x=0; \quad 0 = \sin^2 \theta \Rightarrow \boxed{\theta = 0}$$

$$\text{put } x=1; \quad 1 = \sin^2 \theta \Rightarrow \boxed{\theta = \pi/2}$$

sub in (i),

$$I = \int_{\theta=0}^{\pi/2} \sin^3 \theta \cdot \cos \theta \cdot 2 \sin \theta \cos \theta \, d\theta$$

$$I = \int_{\theta=0}^{\pi/2} 2 \sin^4 \theta \cdot \cos^2 \theta \, d\theta$$

$$\text{WKT } \int_0^{\pi/2} \sin^p \theta \cdot \cos^q \theta \, d\theta = \frac{1}{2} \beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$= \frac{1}{2} \beta\left(\frac{4+1}{2}, \frac{2+1}{2}\right) = \beta\left(\frac{5}{2}, \frac{3}{2}\right)$$

$$I = \frac{\Gamma(5/2) \cdot \Gamma(3/2)}{\Gamma(5/2 + 3/2)}$$

$$\Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2} + 1\right)$$

$$= \frac{1}{2} \Gamma\left(\frac{1}{2}\right)$$

$$= \frac{3}{2} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right) \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right)$$

$$\Gamma\left(\frac{5}{2}\right) = \Gamma\left(\frac{3}{2} + 1\right)$$

$$= \frac{3}{2} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right)$$

$$\Gamma(n+1) = n! \quad \leftarrow \Gamma(4)$$

$$= \frac{3}{8} \frac{\sqrt{\pi} \sqrt{\pi}}{3!} \Rightarrow \boxed{I = \frac{\pi}{16}}$$

$$\Gamma\left(\frac{7}{2}\right) = \Gamma\left(\frac{5}{2} + 1\right)$$

$$= \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right)$$

$$\textcircled{6} \int_0^2 (4-x^2)^{3/2} \, dx$$

$$\text{Let } I = \int_0^2 (4-x^2)^{3/2} \, dx \quad \text{--- (1)}$$

on comparing, $a=4$, $n=2$

$$\boxed{x^2 = 4 \sin^2 \theta}$$

$$\Rightarrow x = 2 \sin \theta$$

$$\Rightarrow dx = 2 \cos \theta \, d\theta$$

consider, $(4-x^2)^{3/2}$

$$= (4 - 4 \sin^2 \theta)^{3/2}$$

$$= 4^{3/2} (1 - \sin^2 \theta)^{3/2}$$

$$= \sqrt{64} \cos^3 \theta \Rightarrow 8 \cos^3 \theta$$

consider, $x = 2 \sin \theta$

$$\text{put } x=0 ; 0 = 2 \sin \theta \Rightarrow \boxed{\theta=0}$$

$$\text{put } x=2 ; 2 = 2 \sin \theta \Rightarrow \boxed{\theta = \frac{\pi}{2}}$$

Sub in ①,

$$I = \int_{\theta=0}^{\pi/2} 8 \cos^3 \theta \cdot 2 \cos \theta d\theta$$

$$= \int_{\theta=0}^{\pi/2} 16 \sin^0 \theta \cdot \cos^4 \theta d\theta$$

$$\text{WKT, } \int_0^{\pi/2} \sin^p \theta \cdot \cos^q \theta \cdot d\theta = \frac{1}{2} \beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$= 16 \cdot \frac{1}{2} \beta\left(\frac{0+1}{2}, \frac{4+1}{2}\right) = 8 \beta\left(\frac{1}{2}, \frac{5}{2}\right)$$

$$= 8 \cdot \frac{\Gamma(1/2) \cdot \Gamma(5/2)}{\Gamma\left(\frac{1}{2} + \frac{5}{2}\right)}$$

$$= \frac{\sqrt{\pi} \cdot \frac{3}{2} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right)}{8 \times \Gamma(3)}$$

$$= \frac{8 \times \sqrt{\pi} \times 3 \times \sqrt{\pi}}{2!} = \frac{2 \times 3 \sqrt{\pi} \sqrt{\pi}}{2} = \boxed{I = 3\pi}$$

⑦

$$\int_0^{\infty} \frac{dx}{\sqrt{x}(1+x)}$$

$$\text{Let } I = \int_0^{\infty} x^{-1/2} \cdot \frac{1}{(1+x)} dx \quad \text{--- ①}$$

For expressions of the form $(a+x^n)$,
we sub $x^n = a \tan^2 \theta$

here $a=1$ $n=1$

$$\therefore \boxed{x = \tan^2 \theta}$$

$$\Rightarrow dx = 2 \tan \theta \sec^2 \theta d\theta$$

$$\text{consider } x^{-1/2} = (\tan^2 \theta)^{-1/2} = \tan^{-1} \theta = \frac{\cos \theta}{\sin \theta}$$

$$\text{and } (1+x) = (1+\tan^2 \theta) = \sec^2 \theta = \frac{1}{\cos^2 \theta}$$

$$dx = 2 \tan \theta \sec^2 \theta d\theta = 2 \frac{\sin \theta}{\cos \theta} \frac{1}{\cos^2 \theta} d\theta$$

$$dx = \frac{2 \sin \theta d\theta}{\cos^3 \theta}$$

consider $x = \tan^2 \theta$

$$\text{put } x = 0 ; 0 = \tan^2 \theta \Rightarrow \boxed{\theta = 0}$$

$$\text{put } x = \infty ; \infty = \tan^2 \theta \Rightarrow \boxed{\theta = \frac{\pi}{2}}$$

Sub in ①,

$$I = \int_0^{\pi/2} \frac{\cos \theta}{\sin \theta} \cdot \cos^2 \theta \cdot \frac{2 \sin \theta}{\cos^3 \theta} d\theta$$

$$= 2 \int_0^{\pi/2} 1 d\theta$$

$$= 2 \left[\theta \right]_0^{\pi/2} \Rightarrow \boxed{I = \pi}$$

$$= 2 \int_0^{\pi/2} \sin^0 \theta \cos^0 \theta d\theta$$

$$\text{WKT, } \int_0^{\pi/2} \sin^p \theta \cos^q \theta d\theta = \frac{1}{2} \beta \left(\frac{p+1}{2}, \frac{q+1}{2} \right)$$

$$= 2 \cdot \frac{1}{2} \beta \left(\frac{0+1}{2}, \frac{0+1}{2} \right) = \beta \left(\frac{1}{2}, \frac{1}{2} \right)$$

$$I = \frac{\Gamma(\frac{1}{2}) \cdot \Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2} + \frac{1}{2})}$$

$$= \frac{\sqrt{\pi} \cdot \sqrt{\pi}}{\Gamma(1)}$$

$$\Gamma(1)$$

$$= \frac{\pi}{0!} \Rightarrow \boxed{I = \pi}$$

(8)

$$ST \quad \beta(m, n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx$$

$$\text{let } I = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx \quad \text{--- (1)}$$

For expression of the form $(a+x^n)$,
we sub $x^n = a \tan^2 \theta$

here $a=1$, $n=1$

$$\therefore \boxed{x = \tan^2 \theta}$$

$$\Rightarrow dx = 2 \tan \theta \sec^2 \theta d\theta$$

$$\text{put } x=0; \tan^2 \theta = 0 \Rightarrow \boxed{\theta=0}$$

$$\text{put } x=\infty; \tan^2 \theta = \infty \Rightarrow \boxed{\theta = \frac{\pi}{2}}$$

Sub in (1),

$$I = \int_{\theta=0}^{\pi/2} \frac{(\tan^2 \theta)^{m-1}}{(1+\tan^2 \theta)^{m+n}} \cdot 2 \tan \theta \sec^2 \theta d\theta$$

$$= \int_{\theta=0}^{\pi/2} \frac{\tan^{2m-2} \theta}{(\sec^2 \theta)^{m+n}} \cdot 2 \tan \theta \sec^2 \theta d\theta$$

$$= 2 \int_{\theta=0}^{\pi/2} \frac{\tan^{2m-2} \theta}{\sec^{2m+2n} \theta} \cdot \tan \theta \sec^2 \theta d\theta$$

$$= 2 \int_{\theta=0}^{\pi/2} \frac{\tan^{2m-1} \theta}{\sec^{2m+2n-2} \theta} d\theta$$

$$= 2 \int_{\theta=0}^{\pi/2} \frac{\sin^{2m-1} \theta}{\cos^{2m-1} \theta} \cdot \cos^{2m+2n-2} \theta d\theta$$

$$= 2 \int_{\theta=0}^{\pi/2} \sin^{2m-1} \theta \cdot \cos^{2m+2n-2-2m+1} \theta d\theta$$

$$= 2 \int_{\theta=0}^{\pi/2} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta$$

$$= \beta(m, n) \quad [\text{by def}^n]$$

$$\text{Thus } \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx = \beta(m, n)$$

$$(8) \int_0^{\infty} \frac{dx}{1+x^4}$$

$$(9) \int_0^1 \frac{\sqrt{1-x}}{x} dx$$

$$\rightarrow \text{Let } I = \int_0^{\infty} \frac{1}{1+x^4} dx$$

For expression of the form $(a+x^n)$,

we sub $x^n = a \tan^2 \theta$

here $a=1$ $n=4$

$$\Rightarrow x^4 = \tan^2 \theta$$

$$dx = \sqrt{\tan \theta}$$

$$dx = \frac{1}{2\sqrt{\tan \theta}} \sec^2 \theta d\theta$$

$$x = \sqrt[4]{\tan \theta} = x^2 = \tan \theta$$

$$\text{put } x=0 \rightarrow \tan \theta = 0 \Rightarrow \boxed{\theta=0}$$

$$x=\infty \rightarrow \tan \theta = \infty \Rightarrow \boxed{\theta=\pi/2}$$

$$I = \int_{\theta=0}^{\pi/2} \frac{1}{1+\tan^2 \theta} \cdot \frac{1}{2\sqrt{\tan \theta}} \sec^2 \theta d\theta$$

$$\frac{1}{2} \int_0^{\pi/2} \cos^{1/2} \theta \sin^{-1/2} \theta d\theta$$

By using formula,

$$= \frac{\beta}{2} \frac{1}{2} \left(\frac{-1/2+1}{2}, \frac{1/2+1}{2} \right) = \frac{1}{4} \beta \left(\frac{1}{4}, \frac{3}{4} \right)$$

$$= \frac{1}{4} \frac{\Gamma(1/4) \Gamma(3/4)}{\Gamma(1/4 + 3/4)}$$

$$= \frac{1}{4} \pi \sqrt{2} \Rightarrow \boxed{I = \frac{\pi}{2\sqrt{2}}}$$

Q →

$$\int_0^1 \sqrt{\frac{1-u}{u}} du$$

$$\int_0^1 \frac{(1-u)^{1/2}}{u^{1/2}} du$$

For expression of the form $(a \mp u^n)$,
we sub $u^n = a \sin^2 \theta$.

Here, $a=1$ $n=1$

$$u = \sin^2 \theta$$

$$du = 2 \sin \theta \cos \theta d\theta$$

$$u^{1/2} = (\sin^2 \theta)^{1/2} \Rightarrow u^{1/2} = \sin \theta$$

$$(1-u)^{1/2} = (1-\sin^2 \theta)^{1/2} = (\cos^2 \theta)^{1/2} = \cos \theta$$

$$\text{put } u=0 ; \boxed{\theta=0}$$

$$u=1, \boxed{\theta=\pi/2}$$

$$I = \int_{\theta=0}^{\pi/2} \frac{\cos \theta}{\sin \theta} \cdot d\theta \cdot 2 \sin \theta \cos \theta$$

$$= 2 \int_{\theta=0}^{\pi/2} \sin^0 \theta \cdot \cos^2 \theta \cdot d\theta$$

$$= \beta \left(\frac{0+1}{2}, \frac{2+1}{2} \right) = \beta \left(\frac{1}{2}, \frac{3}{2} \right)$$

$$= \frac{\Gamma(\frac{1}{2}) \cdot \Gamma(\frac{3}{2})}{\Gamma(\frac{1}{2} + \frac{3}{2})}$$

$$= \frac{\sqrt{\pi} \times \frac{1}{2} \sqrt{\pi}}{2}$$

1

$$\Rightarrow \boxed{I = \frac{\pi}{2}}$$