CALCULUS AND LINEAR ALGEBRA

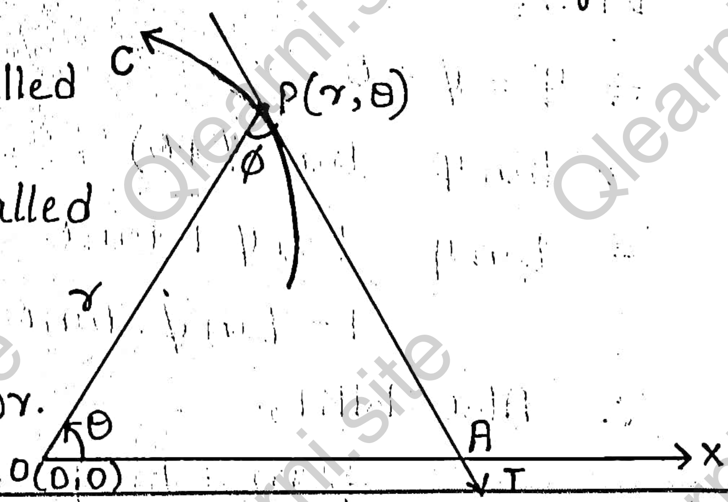
Module - 1

POLAR CURVES AND EVALUATES

Introduction :-

Let 'P' be any point on the plane $O(0,0)$ be an origin, which extends the $\overrightarrow{OX}$ and $OP = r$ and 'C' be a curve at the point 'P' with the tangent 'T'.

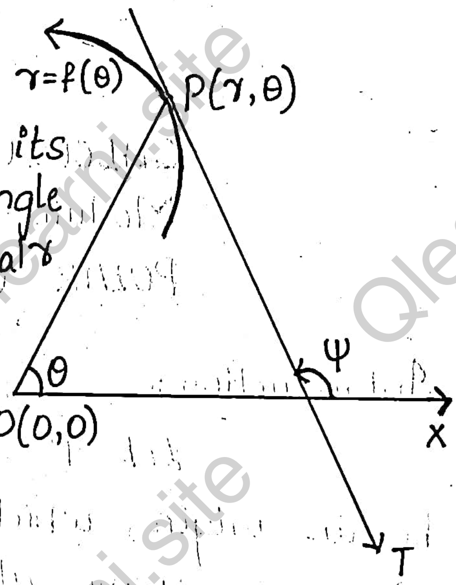
Here, the point $O(0,0)$ is called pole,
the point $P(r,\theta)$ is called polar co-ordinates,
the distance $OP = r$ is called radius of vector.



'θ' be the angle between radius of vector and initial line,
'ϕ' is called the angle between the radius of vector and tangent to the polar curve 'C' and the polar curve which is of the form either $r = f(\theta)$ or $f(r,\theta) = C$.
The entire above system is called "Polar System".

ANGLE BETWEEN THE RADIUS OF VECTOR AND TANGENT TO THE POLAR CURVE :-

Given,
$P(r,\theta)$ be anypoint on the plane,
Let, $OP = r$ be the radius of a vector, which makes the angle to its polar curve 'ϕ' and 'ψ' be the angle made by the tangent to the initial line.
Finally, 'θ' be the angle between radius of vector and initial line $\overrightarrow{OX}$.



WKT,

$$\Rightarrow \psi = \phi + \theta$$

$$\Rightarrow \tan\psi = \tan(\phi + \theta)$$

$$\Rightarrow \tan\psi = \frac{\tan\phi + \tan\theta}{1 - \tan\phi \cdot \tan\theta} \quad \text{——} (1)$$

$\therefore$ Also WKT,

$$\Rightarrow m = \tan\psi = \frac{dy}{dx}$$

$$\Rightarrow \tan\psi = \frac{dy}{dx}$$

$$\Rightarrow r\cos\theta = x \text{ ——— } ③ \qquad y = r\sin\theta \text{ ——— } ④$$

diff w.r to 'θ'

$$③ \Rightarrow \cos\theta \frac{dr}{d\theta} + r(-\sin\theta) = \frac{dx}{d\theta}$$

$$\Rightarrow \cos\theta \frac{dr}{d\theta} - r\sin\theta = \frac{dx}{d\theta}$$

$$④ \Rightarrow \frac{dy}{d\theta} = \sin\theta \frac{dy}{d\theta} + r\cos\theta$$

From equation ②

$$\Rightarrow \tan\psi = \frac{dy/d\theta}{dx/d\theta}$$

$$\Rightarrow \tan\psi = \frac{\sin\theta \cdot \frac{dy}{d\theta} + r\cos\theta}{\cos\theta \cdot \frac{dr}{d\theta} - r\sin\theta}$$

$$\Rightarrow \tan\psi = \frac{\dfrac{\sin\theta \cdot \frac{\cancel{dr}}{\cancel{d\theta}}}{\cos\theta \cdot \frac{\cancel{dr}}{\cancel{d\theta}}} + \dfrac{r\cancel{\cos\theta}}{\cancel{\cos\theta} \cdot \frac{dr}{d\theta}}}{\dfrac{\cancel{\cos\theta} \cdot \frac{dr}{\cancel{d\theta}}}{\cancel{\cos\theta} \cdot \frac{dr}{\cancel{d\theta}}} - \dfrac{r\sin\theta}{\cos\theta \cdot \frac{dr}{d\theta}}}$$

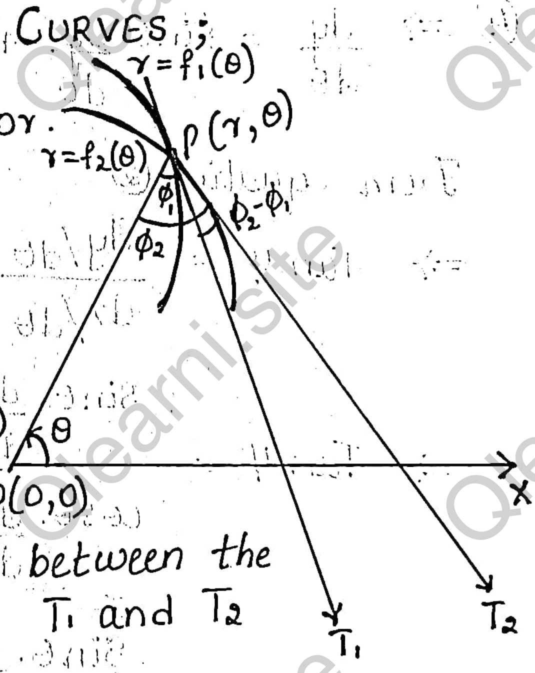
$$\Rightarrow \tan\psi = \frac{\tan\theta + r\left[\frac{d\theta}{dr}\right]}{1 - (\tan\theta)\left[r\,\frac{d\theta}{dr}\right]}$$

$$\Rightarrow \frac{\tan\theta + \tan\phi}{1 - \tan\theta . \tan\phi} = \frac{\tan\theta + r\frac{d\theta}{dr}}{1 - \tan\theta\left[r\frac{d\theta}{dr}\right]}$$

$$\Rightarrow \tan\phi = r\frac{d\theta}{dr} \quad [\text{or}]$$

$$\Rightarrow \cot\phi = \frac{1}{r}\frac{dr}{d\theta}$$

ANGLE BETWEEN TWO POLAR CURVES :



Let, $OP = r$ be the radius of vector. $r = f_1(\theta)$, $r = f_2(\theta)$ be the two polar curves intersecting at 'P'.

Let, T_1 and T_2 be the two tangents to the curves, $r = f_1(\theta)$ and $r = f_2(\theta)$ at 'P'.

Let, ϕ_1 and ϕ_2 be the angle between the radius of vector and tangent T_1 and T_2 respectively.

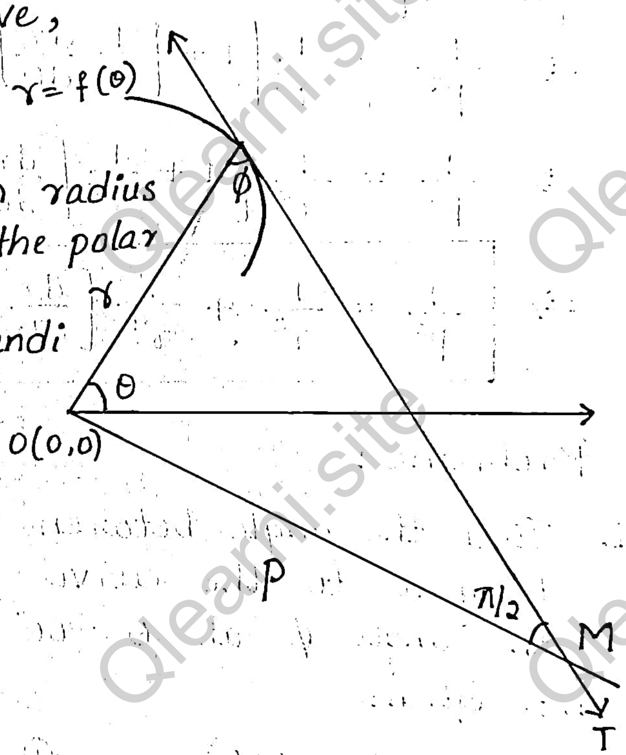
$\therefore$ The angle between the given two polar curves can be evaluated as ϕ_1 and ϕ_2 ...
If $|\phi_2 - \phi_2| = \frac{\pi}{2}$, then we can say that the given curves are interesecting perpendicular (or) Orthrogonally.

PROVE THAT:

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left[\frac{dr}{d\theta}\right]^2$$

using usual notations.

Let $OP = r$, be the radius of vector,
Let $r = f(\theta)$ be a polar curve,
Let 'T' be the tangent to the polar curve at 'P',
Let 'ϕ' be the angle between radius of vector and tangent to the polar curve,
Let 'M' be the foot of perpendicular,
Let $OM = P$,



$\therefore$ From, ΔOPM;

$$\Rightarrow \sin\phi = \frac{OM}{OP}$$

$$\Rightarrow \sin\phi = \frac{P}{r}$$

$$\Rightarrow \boxed{p = r\sin\phi} \text{ — ① [SOBS]}$$

[reciprocal]

$$\Rightarrow p^2 = r^2\sin^2\phi$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2\sin^2\phi}$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\operatorname{cosec}^2\phi$$

$$\Rightarrow \boxed{\frac{1}{p^2} = \frac{1}{r^2}(1+\cot^2\phi)} \text{ — ②}$$

WKT,

$$\Rightarrow \tan\phi = r\frac{d\theta}{dr}$$

$$\Rightarrow \cot\phi = \frac{1}{r}\frac{dr}{d\theta}$$

from equation ②

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\left[1+\left[\frac{1}{r}\frac{dr}{d\theta}\right]^2\right]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\left[1+\frac{1}{r^2}\left[\frac{dr}{d\theta}\right]^2\right]$$

$$\Rightarrow \boxed{\frac{1}{p^2} = \frac{1}{r^2}+\frac{1}{r^4}\left[\frac{dr}{d\theta}\right]^2}$$

Hence proved.

Problems :

1. Find the angle between the radius of vector and tangent to the curve $r = a(1-\cos\theta)$, also find the angle ϕ at $\theta = 60°$

Soln :- Given :-

$$\Rightarrow r = a(1-\cos\theta) \quad \text{——— ①}$$

diff w.r to θ

$$\Rightarrow \frac{dr}{d\theta} = a(0-(-\sin\theta))$$

$$\Rightarrow \frac{dr}{d\theta} = a\sin\theta$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{a\sin\theta}{r}$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{\cancel{a}\sin\theta}{\cancel{a}(1-\cos\theta)}$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{\cancel{2}\,\cancel{\sin\theta/2}\cdot\cos\theta/2}{\cancel{2}\sin^{\cancel{2}}\theta/2}$$

$$\Rightarrow \cot\phi = \cot\left[\frac{\theta}{2}\right]$$

$$\Rightarrow \quad \phi = \frac{\theta}{2}$$

$\therefore$ where, $\theta = \pi/3$

$$\Rightarrow \theta = \frac{\pi/3}{2} \quad \Rightarrow \boxed{\phi = \frac{\pi}{6}}$$

2. Find the angle b/w the radius of vector and tangent to the curve $r = a(1+\cos\theta)$, also find the angle ϕ at $\theta = \pi/6$.

Soln:-

$$\Rightarrow r = a(1+\cos\theta) \quad \text{①}$$

diff w.r to θ

$$\Rightarrow \frac{dr}{d\theta} = -a\sin\theta$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = -\frac{a\sin\theta}{r}$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = -\frac{\cancel{a}\sin\theta}{\cancel{a}(1+\cos\theta)}$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{-\cancel{2}\sin\theta/2\ \cancel{\cos\theta/2}}{\cancel{2}\cos^{\cancel{2}}\theta/2}$$

$$\Rightarrow \cot\phi = -\tan\left[\frac{\theta}{2}\right]$$

$$\Rightarrow \cot\phi = \tan\left[\frac{\pi}{2} + \frac{\theta}{2}\right]$$

$$\Rightarrow \cot\phi = \cot\left[\frac{\pi}{2} + \frac{\theta}{2}\right]$$

$$\Rightarrow \phi = \frac{\pi}{2} + \frac{\theta}{2}$$

Where, $\theta = \frac{\pi}{6} \Rightarrow \phi = \frac{\pi}{2} + \frac{\pi/6}{2}$

$$\Rightarrow \phi = \frac{\pi}{2} + \frac{\pi}{12}$$

$$\Rightarrow \phi = \frac{6\pi + \pi}{12} \Rightarrow \boxed{\phi = \frac{7\pi}{12}}$$

3. Show that the given pairs of polar curves can intersect orthogonally.

i) $r = a(1-\cos\theta)$, $r = b(1+\cos\theta)$

$\Rightarrow r = a(1-\cos\theta)$ —— ①

$\Rightarrow$ diff w.r to θ

$$\Rightarrow \frac{dr}{d\theta} = a(0-(-\sin\theta))$$

$$\Rightarrow \frac{dr}{d\theta} = a\sin\theta$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{a\sin\theta}{r}$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{\cancel{a}\sin\theta}{\cancel{a}(1-\cos\theta)}$$

$$\Rightarrow \cot\phi_1 = \frac{\cancel{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cancel{2}\sin^2\frac{\theta}{2}}$$

$$\Rightarrow \cot\phi_1 = \cot\frac{\theta}{2}$$

$$\Rightarrow \phi_1 = \frac{\theta}{2}$$

$\Rightarrow r = b(1+\cos\theta)$ —— ②

$\Rightarrow$ diff. w.r to θ

$$\Rightarrow \frac{dr}{d\theta} = b(0-\sin\theta)$$

$$\Rightarrow \frac{dr}{d\theta} = -b\sin\theta$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = -\frac{b\sin\theta}{r}$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = -\frac{\cancel{b}\sin\theta}{\cancel{b}(1+\cos\theta)}$$

$$\Rightarrow \cot\phi_2 = \frac{-\cancel{2}\sin\frac{\theta}{2}\cancel{\cos\frac{\theta}{2}}}{\cancel{2}\cos^{\cancel{2}}\frac{\theta}{2}}$$

$$\Rightarrow \cot\phi_2 = -\tan\frac{\theta}{2}$$

$$\Rightarrow \cot\phi_2 = \cot\left[\frac{\theta}{2} + \frac{\pi}{2}\right]$$

$$\Rightarrow \phi_2 = \frac{\theta}{2} + \frac{\pi}{2}$$

$$\Rightarrow |\phi_2 - \phi_1| = \frac{\pi}{2} + \cancel{\frac{\theta}{2}} - \cancel{\frac{\theta}{2}}$$

$$\Rightarrow |\phi_2 - \phi_1| = \frac{\pi}{2}$$

ii) $r = a(1+\sin\theta)$ | $r = b(1-\sin\theta)$

$$\Rightarrow \quad r = a(1+\sin\theta) \quad \text{—} ①$$

diff w.r to θ

$$\Rightarrow \quad \frac{dr}{d\theta} = a\cos\theta$$

$$\Rightarrow \quad \frac{1}{r}\frac{dr}{d\theta} = \frac{a\cos\theta}{r}$$

$$\Rightarrow \quad \frac{1}{r}\frac{dr}{d\theta} = \frac{\cancel{a}\cos\theta}{\cancel{a}(1+\sin\theta)}$$

$$\Rightarrow \quad \frac{1}{r}\frac{dr}{d\theta} = \frac{\cos^2\theta/2 \quad \sin^2\theta/2}{\cos^2\theta/2 + \sin^2\theta/2 + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$\Rightarrow \quad \frac{1}{r}\frac{dr}{d\theta} = \frac{\cancel{\left[\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right]}\left[\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right]}{\left[\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right]^{\cancel{2}}}$$

$$\Rightarrow \quad \frac{1}{r}\frac{dr}{d\theta} = \frac{\cos\theta/2 - \sin\theta/2}{\cos\theta/2 + \sin\theta/2}$$

$$\Rightarrow \quad \frac{1}{r}\frac{dr}{d\theta} = \frac{\frac{\cos\theta/2}{\cos\theta/2} - \frac{\sin\theta/2}{\cos\theta/2}}{\frac{\cos\theta/2}{\cos\theta/2} + \frac{\sin\theta/2}{\cos\theta/2}}$$

$$\Rightarrow \quad \frac{1}{r}\frac{dr}{d\theta} = \frac{1-\tan\theta/2}{1+\tan\theta/2}$$

$$\Rightarrow \quad \frac{1}{r}\frac{dr}{d\theta} = \frac{\tan(\pi/4) - \tan(\theta/2)}{1+\tan(\pi/4)\tan(\theta/2)}$$

$$\Rightarrow \quad \frac{1}{r}\frac{dr}{d\theta} = \tan\left[\frac{\pi}{4} - \frac{\theta}{2}\right]$$

$$\Rightarrow \cot \phi_1 = \cot\left[\frac{\pi}{2} - \left[\frac{\pi}{4} - \frac{\theta}{2}\right]\right]$$

$$\Rightarrow \phi_1 = \frac{\pi}{2} - \frac{\pi}{4} + \frac{\theta}{2}$$

$$\Rightarrow \phi_1 = \frac{\pi}{4} + \frac{\theta}{2}$$

$$r = b(1 - \sin\theta) \text{ —— } (2)$$

$$\Rightarrow \frac{dr}{d\theta} = -b\cos\theta$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = -\frac{b\cos\theta}{r}$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{-\cancel{b}\cos\theta}{\cancel{b}(1-\sin\theta)}$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{-\cos^2[\theta/2]\ \sin^2[\theta/2]}{\cos^2\left[\frac{\theta}{2}\right] + \sin^2\left[\frac{\theta}{2}\right] - 2\sin\left[\frac{\theta}{2}\right]\cos\left[\frac{\theta}{2}\right]}$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{-\left[\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right]\cancel{\left[\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right]}}{\left[\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right]^{\cancel{2}}}$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{-\cos\frac{\theta}{2} + \sin\frac{\theta}{2}}{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}}$$

$$\Rightarrow \cot\phi_2 = -\left|\frac{1 + \tan\theta/2}{1 - \tan\theta/2}\right|$$

$$\Rightarrow \cot\phi_2 = -\tan\left[\frac{\pi}{4} + \frac{\theta}{2}\right]$$

$$\Rightarrow \cot\phi_2 = \cot\left[\frac{\pi}{2} + \left[\frac{\pi}{4} + \frac{\theta}{2}\right]\right]$$

$$\Rightarrow \phi_2 = \frac{\pi}{2} + \frac{\pi}{4} + \frac{\theta}{2}$$

$$\Rightarrow |\phi_2 - \phi_1| = \left|\frac{\pi}{2} + \cancel{\frac{\pi}{4}} + \cancel{\frac{\theta}{2}} - \cancel{\frac{\pi}{4}} - \cancel{\frac{\theta}{2}}\right|$$

$$\Rightarrow |\phi_2 - \phi_1| = \frac{\pi}{2}$$

iii) $r = \dfrac{a}{1+\cos\theta}$

$\Rightarrow r(1+\cos\theta) = a$ —— ①

$\Rightarrow$ diff w.r to θ

$\Rightarrow (1+\cos\theta)\dfrac{dr}{d\theta} + r(-\sin\theta) = 0$

$\Rightarrow (1+\cos\theta)\dfrac{dr}{d\theta} = r\sin\theta$

$\Rightarrow \dfrac{1}{r}\dfrac{dr}{d\theta} = \dfrac{\sin\theta}{1+\cos\theta}$

$\Rightarrow \dfrac{1}{r}\dfrac{dr}{d\theta} = \dfrac{\cancel{2}\sin\theta/2\,\cancel{\cos\theta/2}}{\cancel{2}\cos^2\theta/2}$

$\Rightarrow \dfrac{1}{r}\dfrac{dr}{d\theta} = \dfrac{\sin\theta/2}{\cos\theta/2}$

$\Rightarrow \cot\phi_1 = \tan\dfrac{\theta}{2}$

$\Rightarrow \cot\phi_1 = \cot\left[\dfrac{\pi}{2} - \dfrac{\theta}{2}\right]$

$\Rightarrow \phi_1 = \dfrac{\pi}{2} - \dfrac{\theta}{2}$

$r = \dfrac{b}{1-\cos\theta}$

$\Rightarrow r(1-\cos\theta) = b$ —— ②

$\Rightarrow$ diff w.r to θ

$\Rightarrow (1-\cos\theta)\dfrac{dr}{d\theta} + r(\sin\theta) = 0$

$\Rightarrow (1-\cos\theta)\dfrac{dr}{d\theta} = -r\sin\theta$

$\Rightarrow \dfrac{1}{r}\dfrac{dr}{d\theta} = -\dfrac{\sin\theta}{1-\cos\theta}$

$\Rightarrow \dfrac{1}{r}\dfrac{dr}{d\theta} = \dfrac{-\cancel{2}\cancel{\sin\theta/2}\cos\theta/2}{\cancel{2}\sin^2\theta/2}$

$\Rightarrow \dfrac{1}{r}\dfrac{dr}{d\theta} = -\cot\dfrac{\theta}{2}$

$\Rightarrow \cot\phi_2 = \cot\left[-\dfrac{\theta}{2}\right]$

$\Rightarrow \phi_2 = -\dfrac{\theta}{2}$

$$\Rightarrow |\phi_2 - \phi_1| = -\frac{\theta}{2} - \left[\frac{\pi}{2} - \frac{\theta}{2}\right]$$

$$\Rightarrow |\phi_2 - \phi_1| = -\cancel{\frac{\theta}{2}} - \frac{\pi}{2} + \cancel{\frac{\theta}{2}}$$

$$\Rightarrow |\phi_2 - \phi_1| = \frac{\pi}{2}$$

iv) $r^n = a^n \cos n\theta$

$\Rightarrow r^n = a^n \cos n\theta$ ——— ①

diff w.r to θ

$\Rightarrow \cancel{n} r^{n-1} \dfrac{dr}{d\theta} = a^n(-\sin n\theta)(\cancel{n})$

$\Rightarrow \dfrac{r^n}{r} \dfrac{dr}{d\theta} = -a^n \sin n\theta$

$\Rightarrow \dfrac{1}{r} \dfrac{dr}{d\theta} = -\dfrac{a^n \sin n\theta}{r^n}$

$\Rightarrow \dfrac{1}{r} \dfrac{dr}{d\theta} = -\dfrac{\cancel{a^n} \sin n\theta}{\cancel{a^n} \cos n\theta}$

$\Rightarrow \dfrac{1}{r} \dfrac{dr}{d\theta} = -\tan n\theta$

$\Rightarrow \cot \phi_1 = \cot\left[\dfrac{\pi}{2} + n\theta\right]$

$\Rightarrow \phi_1 = \dfrac{\pi}{2} + n\theta$

$r^n = b^n \sin n\theta$

$\Rightarrow r^n = b^n \sin n\theta$ ——— ②

diff. w.r to 'θ'

$\Rightarrow \cancel{n} r^{n-1} \dfrac{dr}{d\theta} = b^n(\cos n\theta)(\cancel{n})$

$\Rightarrow \dfrac{r^n}{r} \dfrac{dr}{d\theta} = b^n \cos n\theta$

$\Rightarrow \dfrac{1}{r} \dfrac{dr}{d\theta} = \dfrac{b^n \cos n\theta}{r^n}$

$\Rightarrow \dfrac{1}{r} \dfrac{dr}{d\theta} = \dfrac{\cancel{b^n} \cos n\theta}{\cancel{b^n} \sin n\theta}$

$\Rightarrow \dfrac{1}{r} \dfrac{dr}{d\theta} = \cot n\theta$

$\Rightarrow \cot \phi_2 = \cot n\theta$

$\Rightarrow \phi_2 = n\theta$

$\Rightarrow |\phi_2 - \phi_1| = \cancel{n\theta} - \dfrac{\pi}{2} - \cancel{n\theta}$

$\Rightarrow |\phi_2 - \phi_1| = \dfrac{\pi}{2}$

v) $r^n \cos n\theta = a^n$

$\Rightarrow r^n = \dfrac{a^n}{\cos n\theta}$ ——— ①

$\Rightarrow$ diff w.r to θ

$\Rightarrow \cancel{n} r^{n-1} \dfrac{dr}{d\theta} = \dfrac{0(\cos n\theta) - a^n(-\sin n\theta)(\cancel{n})}{(\cos n\theta)^2}$

$\Rightarrow \dfrac{r^n}{r} \dfrac{dr}{d\theta} = \dfrac{a^n \sin n\theta}{(\cos n\theta)^2}$

$\Rightarrow \dfrac{1}{r} \dfrac{dr}{d\theta} = \dfrac{a^n \sin n\theta}{(\cos n\theta)^2 r^n}$

$r^n \sin n\theta = b^n$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{\cancel{a^n}\sin n\theta}{(\cos n\theta)^{\cancel{2}}\left[\frac{\cancel{a^n}}{\cancel{\cos n\theta}}\right]}$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{\sin n\theta}{\cos n\theta}$$

$$\Rightarrow \cot\phi_1 = \tan n\theta$$

$$\Rightarrow \cot\phi_1 = \cot\left[\frac{\pi}{2} - n\theta\right]$$

$$\Rightarrow \phi_1 = \frac{\pi}{2} - n\theta$$

$$r^n \sin n\theta = b^n \quad \text{——— (2)}$$

$$\Rightarrow r^n = \frac{b^n}{\sin n\theta}$$

diff w.r to θ

$$\Rightarrow \cancel{n}r^{n-1}\frac{dr}{d\theta} = \frac{0(\sin n\theta) - b^n(\cos n\theta)\cancel{(n)}}{(\sin n\theta)^2}$$

$$\Rightarrow \frac{r^n}{r}\frac{dr}{d\theta} = -\frac{b^n\cos n\theta}{(\sin n\theta)^2}$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = -\frac{b^n\cos n\theta}{(\sin n\theta)^2 r^n}$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = -\frac{\cancel{b^n}\cos n\theta}{(\sin n\theta)^{\cancel{2}}\left[\frac{\cancel{b^n}}{\cancel{\sin n\theta}}\right]}$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = -\frac{\cos n\theta}{\sin n\theta}$$

$$\Rightarrow \cot\phi_2 = -\cot n\theta$$

$$\Rightarrow \cot\phi_2 = \cot(-n\theta)$$

$$\Rightarrow \phi_2 = -n\theta$$

$$\Rightarrow |\phi_2 - \phi_1| = -\cancel{n\theta} - \frac{\pi}{2} + \cancel{n\theta}$$

$$\Rightarrow |\phi_2 - \phi_1| = \frac{\pi}{2}$$

vi) $r = 4\sec^2\left(\frac{\theta}{2}\right)$

$\Rightarrow r = \dfrac{4}{\cos^2\left(\frac{\theta}{2}\right)}$

$\Rightarrow r\cos^2\left(\frac{\theta}{2}\right) = 4$

$\Rightarrow \dfrac{r(1+\cos\theta)}{2} = 4$

$\Rightarrow r(1+\cos\theta) = 8$ —— ①

diff w.r to θ

$\Rightarrow (1+\cos\theta)\dfrac{dr}{d\theta} + r(-\sin\theta) = 0$

$\Rightarrow (1+\cos\theta)\dfrac{dr}{d\theta} = r\sin\theta$

$\Rightarrow \dfrac{1}{r}\dfrac{dr}{d\theta} = \dfrac{\sin\theta}{1+\cos\theta}$

$\Rightarrow \dfrac{1}{r}\dfrac{dr}{d\theta} = \dfrac{\cancel{2}\sin\frac{\theta}{2}\cancel{\cos\frac{\theta}{2}}}{\cancel{2}\cos^{\cancel{2}}\frac{\theta}{2}}$

$\Rightarrow \cot\phi_1 = \tan\left[\frac{\theta}{2}\right]$

$\Rightarrow \cot\phi_1 = \cot\left[\frac{\pi}{2} - \frac{\theta}{2}\right]$

$\Rightarrow \phi_1 = \frac{\pi}{2} - \frac{\theta}{2}$

$r = 9\,\mathrm{cosec}^2\left[\theta/2\right]$

$\Rightarrow r = \dfrac{9}{\sin^2\left(\frac{\theta}{2}\right)}$

$\Rightarrow r\sin^2\left[\frac{\theta}{2}\right] = 9$

$\Rightarrow \dfrac{r(1-\cos\theta)}{2} = 9$

$\Rightarrow r(1-\cos\theta) = 18$ —— ②

diff. w.r to θ

$\Rightarrow (1-\cos\theta)\dfrac{dr}{d\theta} + r\sin\theta = 0$

$\Rightarrow (1-\cos\theta)\dfrac{dr}{d\theta} = -r\sin\theta$

$\Rightarrow \dfrac{1}{r}\dfrac{dr}{d\theta} = -\dfrac{\sin\theta}{1-\cos\theta}$

$\Rightarrow \dfrac{1}{r}\dfrac{dr}{d\theta} = -\dfrac{\cancel{2}\cancel{\sin\theta/2}\cos\theta/2}{\cancel{2}\sin^{\cancel{2}}\theta/2}$

$\Rightarrow \cot\phi_2 = -\cot(\theta/2)$

$\Rightarrow \cot\phi_2 = \cot\left[-\frac{\theta}{2}\right]$

$\Rightarrow \phi_2 = -\frac{\theta}{2}$

$\Rightarrow |\phi_2 - \phi_1| = \cancel{-\frac{\theta}{2}} - \frac{\pi}{2} + \cancel{\frac{\theta}{2}}$

$\Rightarrow |\phi_2 - \phi_1| = \frac{\pi}{2}$

4. Find the angle between the given polar curves;

i) $r = 2\sin\theta$

$\Rightarrow r = 2\sin\theta$ —— ①

diff w.r to θ

$\Rightarrow \dfrac{dr}{d\theta} = 2\cos\theta$

$\Rightarrow \dfrac{1}{r}\dfrac{dr}{d\theta} = \dfrac{2\cos\theta}{r}$

$\Rightarrow \cot\phi_1 = \dfrac{\cancel{2}\cos\theta}{\cancel{2}\sin\theta}$

$\Rightarrow \cot\phi_1 = \cot\theta$

$\Rightarrow \phi_1 = \theta$

$r = 2\cos\theta$

$\Rightarrow r = 2\cos\theta$ —— ②

diff w.r to θ

$\Rightarrow \dfrac{dr}{d\theta} = -2\sin\theta$

$\Rightarrow \dfrac{1}{r}\dfrac{dr}{d\theta} = \dfrac{-2\sin\theta}{r}$

$\Rightarrow \cot\phi_2 = -\dfrac{\cancel{2}\sin\theta}{\cancel{2}\cos\theta}$

$\Rightarrow \cot\phi_2 = -\tan\theta$

$\Rightarrow \phi_2 = \dfrac{\pi}{2} + \theta$

$\Rightarrow |\phi_2 - \phi_1| = \dfrac{\pi}{2} + \cancel{\theta} - \cancel{\theta}$

$\Rightarrow |\phi_2 - \phi_1| = \dfrac{\pi}{2}$

ii) $r = a\log\theta$

$\Rightarrow r = a\log\theta$ —— ①

$\Rightarrow$ diff w.r to 'θ'

$\Rightarrow \dfrac{dr}{d\theta} = a\cdot\dfrac{1}{\theta}$

$\Rightarrow \dfrac{1}{r}\dfrac{dr}{d\theta} = \dfrac{a}{r\theta}$

$\Rightarrow \dfrac{1}{r}\dfrac{dr}{d\theta} = \dfrac{\cancel{a}}{\cancel{a}\log\theta(\theta)}$

$\Rightarrow r\dfrac{d\theta}{dr} = \theta\log\theta$

$\Rightarrow \tan\phi_1 = \theta\log\theta$ —— ③

$r = \dfrac{a}{\log\theta}$

$\Rightarrow r\log\theta = a$ —— ②

diff w.r to 'θ'

$\Rightarrow (\log\theta)\dfrac{dr}{d\theta} + r\left[\dfrac{1}{\theta}\right] = 0$

$\Rightarrow \log\theta\dfrac{dr}{d\theta} = -\dfrac{r}{\theta}$

$\Rightarrow \dfrac{1}{r}\dfrac{dr}{d\theta} = \dfrac{1}{-\theta\log\theta}$

$\Rightarrow r\dfrac{d\theta}{dr} = -\theta\log\theta$

$\Rightarrow \tan\phi_2 = -\theta\log\theta$ —— ④

WKT,

$$\Rightarrow \tan(\phi_1 - \phi_2) = \frac{\tan\phi_1 - \tan\phi_2}{1 + \tan\phi_1 \cdot \tan\phi_2}$$

$$\Rightarrow \tan(\phi_1 - \phi_2) = \frac{\theta\log\theta - (-\theta\log\theta)}{1 + \theta\log\theta(-\theta\log\theta)}$$

$$\Rightarrow \tan(\phi_1 - \phi_2) = \frac{\theta\log\theta + \theta\log\theta}{1 + \theta^2(\log\theta)^2}$$

$$\Rightarrow \tan(\phi_1 - \phi_2) = \frac{2\theta\log\theta}{1 - \theta^2(\log\theta)^2} \quad \text{——— (4)}$$

From equation (1) & (2)

$$\Rightarrow \alpha\log\theta = \frac{\alpha}{\log\theta}$$

$$\Rightarrow \log\theta = \frac{1}{\log\theta}$$

$$\Rightarrow (\log\theta)^2 = 1$$

$$\Rightarrow \log_e\theta = 1$$

$$\Rightarrow \theta = e^1$$

$$\Rightarrow \theta = e$$

from equation (4)

$$\Rightarrow \tan(\phi_1 - \phi_2) = \frac{2e \cdot 1}{1 - e^2(1)^2}$$

$$\Rightarrow \tan(\phi_1 - \phi_2) = \frac{2e}{1 - e^2}$$

$$\Rightarrow |\phi_1 - \phi_2| = \tan^{-1}\left[\frac{2e}{1-e^2}\right] = 2\tan^{-1}e.$$

iii) $r^2 \sin 2\theta = 4$ $\qquad$ $r^2 = 16\sin 2\theta$

$\Rightarrow r^2 \sin 2\theta = 4$ —— ①

diff w r to θ

$\Rightarrow 2r\frac{dr}{d\theta}\sin 2\theta + r^2\cos 2\theta(2) = 0$

$\Rightarrow r\frac{dr}{d\theta}\sin 2\theta = -r^2\cos 2\theta$

$\Rightarrow \frac{\cancel{r}}{r^2}\frac{dr}{d\theta} = \frac{-\cos 2\theta}{\sin 2\theta}$

$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = -\cot 2\theta$

$\Rightarrow \cot\phi_1 = \cot(-2\theta)$

$\Rightarrow \phi_1 = -2\theta$

$\Rightarrow r^2 = 16\sin 2\theta$ —— ②

diff w.r to 'θ'

$\Rightarrow 2r\frac{dr}{d\theta} = 16\cos 2\theta(2)$

$\Rightarrow r\frac{dr}{d\theta} = 16\cos 2\theta$

$\Rightarrow \frac{r}{r^2}\frac{dr}{d\theta} = \frac{16\cos 2\theta}{r^2}$

$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{16\cos 2\theta}{16\sin 2\theta}$

$\Rightarrow \cot\phi_2 = \cot 2\theta$

$\Rightarrow \phi_2 = 2\theta$

$\Rightarrow |\phi_2 - \phi_1| = |2\theta - (-2\theta)|$

$\Rightarrow |\phi_2 - \phi_1| = 4\theta$ —— ③

From ① & ②

$\Rightarrow \frac{\cancel{4}}{\sin 2\theta} = \overset{4}{\cancel{16}}\sin 2\theta$

$\Rightarrow 1 = 4\sin^2 2\theta$

$\Rightarrow \sin^2 2\theta = \frac{1}{4}$ $(\div 2)$

$\Rightarrow \sin 2\theta = \frac{1}{2}$

$\Rightarrow 2\theta = \sin^{-1}\left[\frac{1}{2}\right]$

$\Rightarrow 2\theta = \pi/6$

$\Rightarrow \theta = \pi/12$

$\therefore |\phi_2 - \phi_1| = 4\left[\frac{\pi}{12}\right] = \frac{\pi}{3}$

iv) $r = a(1-\cos\theta)$

$\Rightarrow r = a(1-\cos\theta)$ —— ①

diff w.r to 'θ'

$\Rightarrow \dfrac{dr}{d\theta} = a\sin\theta$

$\Rightarrow \dfrac{1}{r}\dfrac{dr}{d\theta} = \dfrac{a\sin\theta}{r}$

$\Rightarrow \dfrac{1}{r}\dfrac{dr}{d\theta} = \dfrac{\cancel{a}\sin\theta}{\cancel{a}(1-\cos\theta)}$

$\Rightarrow \dfrac{1}{r}\dfrac{dr}{d\theta} = \dfrac{\cancel{2}\sin\theta/2\cos\theta/2}{\cancel{2}\sin^2\theta/2}$

$\Rightarrow \cot\phi_1 = \cot\theta/2$

$\Rightarrow \phi_1 = \dfrac{\theta}{2}$

$r = 2a\cos\theta$

$\Rightarrow r = 2a\cos\theta$ —— ②

diff w.r to θ

$\Rightarrow \dfrac{dr}{d\theta} = -2a\sin\theta$

$\Rightarrow \dfrac{1}{r}\dfrac{dr}{d\theta} = -\dfrac{2a\sin\theta}{r}$

$\Rightarrow \dfrac{1}{r}\dfrac{dr}{d\theta} = \dfrac{-\cancel{2a}\sin\theta}{\cancel{2a}\cos\theta}$

$\Rightarrow \dfrac{1}{r}\dfrac{dr}{d\theta} = -\dfrac{\sin\theta}{\cos\theta}$

$\Rightarrow \cot\phi_2 = -\tan\theta$

$\Rightarrow \cot\phi_2 = \cot\left[\dfrac{\pi}{2}+\theta\right]$

$\Rightarrow \phi_2 = \dfrac{\pi}{2}+\theta$

$\Rightarrow |\phi_2-\phi_1| = \dfrac{\pi}{2}+\theta-\dfrac{\theta}{2}$

$\Rightarrow |\phi_2-\phi_1| = \dfrac{\pi}{2}+\dfrac{\theta}{2}$ —— ③

From, ① & ②

$\Rightarrow \cancel{a}(1-\cos\theta) = 2\cancel{a}\cos\theta$

$\Rightarrow 1-\cos\theta = 2\cos\theta$

$\Rightarrow 1 = 2\cos\theta + \cos\theta$

$\Rightarrow 1 = 3\cos\theta$

$\Rightarrow \dfrac{1}{3} = \cos\theta$

$\Rightarrow \theta = \cos^{-1}\left[\dfrac{1}{3}\right]$

$\therefore |\phi_2-\phi_1| = \dfrac{\pi}{2} + \dfrac{\cos^{-1}[1/3]}{2}$

PEDAL EQUATION OF POLAR CURVES :-

Let 'r' = $f(\theta)$ be a polar curve,

Let 'ϕ' be the angle between radius of vector and tangent to the curve $f(\theta)$ and

'p' be the perpendicular distance from pole to the tangent 'T', then

$$p = r\sin\phi \quad (\text{or}) \quad \frac{1}{p^2} = \frac{1}{r^2}(1+\cot^2\phi) \quad (\text{or})$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left[\frac{dr}{d\theta}\right]^2$$

is called the pedal equation of a polar curve $r = f(\theta)$, it is called $p-r$ equation excluding 'θ'.

Problems :-

1. Find the pedal equation of a polar curve.
if $r = a(1+\cos\theta)$

$\Rightarrow \quad r = a(1+\cos\theta)$ ——— (1)

diff w.r to 'θ'

$$\Rightarrow \quad \frac{dr}{d\theta} = -a\sin\theta$$

$$\Rightarrow \quad \frac{1}{r}\frac{dr}{d\theta} = -\frac{a\sin\theta}{r}$$

$$\Rightarrow \quad \frac{1}{r}\frac{dr}{d\theta} = -\frac{\cancel{a}\sin\theta}{\cancel{a}(1+\cos\theta)}$$

$$\Rightarrow \quad \cot\phi = \frac{-\cancel{2}\sin\theta/2\,\cancel{\cos\theta/2}}{\cancel{2}\cos^{\cancel{2}}\theta/2}$$

$$\Rightarrow \quad \cot\phi = -\tan\theta/2$$

WKT,

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\left[1+\cot^2\phi\right]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\left[1+\tan^2\frac{\theta}{2}\right]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\sec^2\frac{\theta}{2}$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2\cos^2\frac{\theta}{2}}$$

$$\Rightarrow p^2 = r^2\cos^2\theta/2$$

$$\Rightarrow p^2 = r^2\left[\frac{1+\cos\theta}{2}\right]$$

$$\Rightarrow p^2 = \frac{r^2}{2}\left[1+\cos\theta\right]$$

$$\Rightarrow p^2 = \frac{r^2}{2}\left[\frac{r}{a}\right]$$

$$\Rightarrow p^2 = \frac{r^3}{2a}$$

$$\Rightarrow 2ap^2 = r^3$$

iii) $r = a(1-\cos\theta)$ —— ②

diff w.r to θ

$$\Rightarrow \frac{dr}{d\theta} = a\sin\theta$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{a\sin\theta}{r}$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{\cancel{a}\sin\theta}{\cancel{a}(1-\cos\theta)}$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{\cancel{2}\cancel{\sin\theta/2}\cos\theta/2}{\cancel{2}\sin^{\cancel{2}}\theta/2}$$

$$\Rightarrow \cot\phi = \cot\theta/2$$

WKT,

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\left[1+\cot^2\phi\right]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\left[1+\cot^2 \theta/2\right]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\left[\operatorname{cosec}^2 \theta/2\right]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2 \sin^2 \theta/2}$$

$$\Rightarrow p^2 = r^2 \sin^2 \theta/2$$

$$\Rightarrow p^2 = r^2\left[\frac{1-\cos\theta}{2}\right]$$

$$\Rightarrow p^2 = \frac{r^2}{2}\left[1-\cos\theta\right]$$

$$\Rightarrow p^2 = \frac{r^2}{2}\left[\frac{r}{a}\right]$$

$$\Rightarrow p^2 = \frac{r^3}{2a}$$

$$\Rightarrow 2ap^2 = r^3$$

iii) $r^m = a^m\left[\cos m\theta + \sin m\theta\right]$

$\Rightarrow$ diff w.r to 'θ'

$$\Rightarrow m r^{m-1}\frac{dr}{d\theta} = a^m\left[-m\sin m\theta + m\cos m\theta\right]$$

$$\Rightarrow \cancel{m}\frac{r^m}{r}\frac{dr}{d\theta} = \cancel{m}\left[\cos m\theta - \sin m\theta\right]a^m$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{(\cos m\theta - \sin m\theta)a^m}{r^m}$$

$$\Rightarrow \cot\phi = \frac{(\cos m\theta - \sin m\theta)\cancel{a^m}}{\cancel{a^m}\left[\cos m\theta + \sin m\theta\right]}$$

$$\Rightarrow 1+\cot^2\phi = 1 + \frac{(\cos m\theta - \sin m\theta)^2}{(\cos m\theta + \sin m\theta)^2}$$

$$\Rightarrow 1+\cot^2\phi = \frac{(\cos m\theta + \sin m\theta)^2 + (\cos m\theta - \sin m\theta)^2}{(\cos m\theta + \sin m\theta)^2}$$

$$\Rightarrow 1+\cot^2\phi = \frac{2(\cos^2 m\theta + \sin^2 m\theta)}{\left[\frac{r^m}{a^m}\right]^2}$$

$$\Rightarrow 1+\cot^2\phi = \frac{2}{\frac{r^{2m}}{a^{2m}}}$$

$$\Rightarrow 1+\cot^2\phi = \frac{2a^{2m}}{r^{2m}}$$

$\therefore$ WKT,

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\left[1+\cot^2\phi\right]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\,\frac{2a^{2m}}{r^{2m}}$$

$$\Rightarrow \frac{1}{p^2} = \frac{2a^{2m}}{r^{2m+2}}$$

$$\Rightarrow 2a^{2m}p^2 = r^{2(m+1)}$$

iv) $r = a(e^{\theta\cot\alpha})$

$$\Rightarrow r = ae^{\theta\cot\alpha} \quad \text{--- (1)}$$

diff w.r to 'θ'

$$\Rightarrow \frac{dr}{d\theta} = a.e^{\theta\cot\alpha}.\cot\alpha$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{(a\cot\alpha)e^{\theta\cot\alpha}}{r}$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{(\alpha\cot\alpha)\cancel{e^{\theta\cot\alpha}}}{\alpha\cancel{e^{\theta\cot\alpha}}}$$

$$\Rightarrow \cot\phi = \cot\alpha$$

$$\Rightarrow \phi = \alpha$$

WKT,

$$p = r\sin\phi$$

$$\boxed{p = r\sin\alpha}$$

v) $\frac{l}{r} = 1 + e\cos\theta$

$$\Rightarrow r(1+e\cos\theta) = l \quad \text{—— (2)}$$

diff w r to 'θ'

$$\Rightarrow (1+e\cos\theta)\frac{dr}{d\theta} + r(0 - e\sin\theta) = 0$$

$$\Rightarrow (1+e\cos\theta)\frac{dr}{d\theta} = re(\sin\theta)$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{e\sin\theta}{1+e\cos\theta}$$

$$\Rightarrow 1+\cot^2\phi = \frac{1+(e\sin\theta)^2}{(1+e\cos\theta)^2}$$

$$\Rightarrow 1+\cot^2\phi = \frac{(1+e\cos\theta)^2 + (e^2\sin^2\theta)}{(1+e\cos\theta)^2}$$

$$\Rightarrow 1+\cot^2\phi = \frac{1+2e\cos\theta + e^2\cos^2\theta + e^2\sin^2\theta}{(1+e\cos\theta)^2}$$

$$\Rightarrow 1+\cot^2\phi = \frac{1+2e\cos\theta + e^2(\cos^2\theta + \sin^2\theta)}{(1+e\cos\theta)^2}$$

$$\Rightarrow 1+\cot^2\phi = \frac{1+2e\cos\theta + e^2}{(1+e\cos\theta)^2}$$

$$\Rightarrow 1+\cot^2\phi = \frac{1+e^2+2\left[\frac{l}{r}-1\right]}{(l/r)^2}$$

$$\Rightarrow 1+\cot^2\phi = \frac{1+e^2+2\frac{l}{r}-2}{l^2/r^2}$$

$$\Rightarrow 1+\cot^2\phi = \frac{r^2}{l^2}\left[e^2+\frac{2l}{r}-1\right]$$

$\therefore$ WKT

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}(1+\cot^2\phi)$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\frac{r^2}{l^2}\left[e^2+\frac{2l}{r}-1\right]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{l^2}\left[e^2+\frac{2l}{r}-1\right]$$

vi) $r^m = a^m\cos m\theta + b^m\sin m\theta$

$\Rightarrow$ diff w.r.to 'θ'

$$\Rightarrow m r^{m-1}\frac{dr}{d\theta} = -ma^m\sin m\theta + mb^m\cos m\theta$$

$$\Rightarrow \frac{mr^m}{r}\frac{dr}{d\theta} = a^m\left[b^m\cos m\theta - a^m\sin m\theta\right]$$

$$\Rightarrow \frac{r^m}{r}\frac{dr}{d\theta} = \frac{b^m\cos m\theta - a^m\sin m\theta}{r^m}$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{b^m\cos m\theta - a^m\sin m\theta}{a^m\cos m\theta + b^m\sin m\theta}$$

$$\Rightarrow \cot^2\phi = \frac{(b^m\cos m\theta - a^m\sin m\theta)^2}{(a^m\cos m\theta + b^m\sin m\theta)^2}$$

$$\Rightarrow 1+\cot^2\phi = \frac{1+(b^m\cos m\theta - a^m\sin m\theta)^2}{(a^m\cos n\theta + b^m\sin m\theta)^2}$$

$$\Rightarrow 1+\cot^2\phi = \frac{(a^m \cos m\theta + b^m \sin m\theta)^2 + (b^m \cos m\theta - a^m \sin m\theta)^2}{(a^m \cos m\theta + b^m \sin m\theta)^2}$$

$$\Rightarrow 1+\cot^2\phi = \frac{a^{2m}\cos^2 n\theta + b^{2m}\sin^2 n\theta + 2a^m b^m \cos n\theta \sin n\theta + b^{2m}\cos^2 n\theta + a^{2m}\sin^2\theta - 2a^m b^m \cos n\theta \sin n\theta}{(r^m)^2}$$

$$\Rightarrow 1+\cot^2\phi = \frac{(a^{2m}+b^{2m})\cos^2 n\theta + (a^{2m}+b^{2m})\sin^2 n\theta}{r^{2m}}$$

$$\Rightarrow 1+\cot^2\phi = \frac{(a^{2m}+b^{2m})(\cos^2 n\theta + \sin^2 n\theta)}{r^{2m}}$$

$$\Rightarrow 1+\cot^2\phi = \frac{a^{2m}+b^{2m}}{r^{2m}}$$

WKT

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\left[1+\cot^2\phi\right]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\left[\frac{a^{2m}+b^{2m}}{r^{2m}}\right]$$

ARC LENGTH OF DIFFERENT CURVES :-

(1) Let $f(x,y)=C$ be a cartesian curve, then the arc length can be evaluated by the given formula.

$$\frac{ds}{dx} = \sqrt{1+\left[\frac{dy}{dx}\right]^2} = \sqrt{1+y'^2} = \sqrt{1+y_1^2}$$

$$\frac{ds}{dy} = \sqrt{1+\left[\frac{dx}{dy}\right]^2} = \sqrt{1+x'^2} = \sqrt{1+x_1^2}$$

(2) Let $x = x(t)$, $y = y(t)$ be the parametric equation for x and y, then

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$\frac{ds}{dt} = \sqrt{[x'(t)]^2 + [y'(t)]^2}$$

(3) Let $f(r, \theta) = c$ be a polar curve, then

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left[\frac{dr}{d\theta}\right]^2}$$

$$\frac{ds}{d\theta} = \sqrt{1 + r^2 \left[\frac{d\theta}{dr}\right]^2}$$

CURVATURE AND RADIUS OF CURVATURE :-

The rate of change of direction of a curve with respect to distance along the curve is called curvature.

It can be denoted by 'k'.

The reciprocal of the curvature is called the radius of curvature.

It can be denoted by 'ρ'.

$\therefore$ The radius of Curvature 'ρ' $= \frac{1}{K} = \left|\frac{ds}{d\psi}\right|$

Derivation of Radius of Curvature in cartesian form;

Let $f(x,y)=c$, curve in cartesian form,
Let 'T' be a tangent to the curve $f(x,y)=c$,
having the slope $m=\tan\psi$

$\Rightarrow f(x,y)=c$

$\Rightarrow m=\tan\psi=y_1$

$\Rightarrow \psi=\tan^{-1}(y_1)$ —— (1)

diff. w.r to θ

$$\Rightarrow \frac{d\psi}{d\theta}=\frac{1}{1+y_1^2}\frac{d(y_1)}{dx}$$

$$\Rightarrow \frac{d\psi}{d\theta}=\frac{y_2}{1+y_1^2}$$

WKT

$$\frac{ds}{dx}=\sqrt{1+y_1^2}$$

also, WKT

$$\Rightarrow \rho=\frac{ds}{d\psi}$$

$$\Rightarrow \rho=\frac{ds/dx}{d\psi/dx}$$

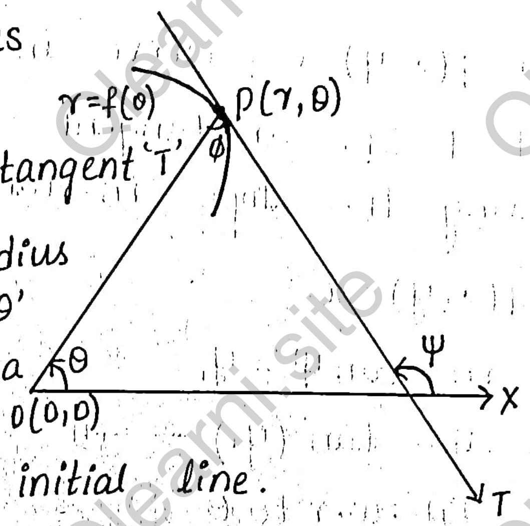
$$\Rightarrow \rho=\frac{\sqrt{1+y_1^2}}{y_2/1+y_1^2}$$

$$\Rightarrow \rho=\frac{(1+y_1^2)^{1/2}(1+y_1^2)^{1}}{y_2}$$

$$\Rightarrow \rho=\frac{(1+y_1^2)^{3/2}}{y_2}, \quad y_2\neq 0$$

Derivative of Radius of Curvature in Polar form :-

Let, $r = f(\theta)$ be a polar curves at $P(r, \theta)$ with $OP = r$ radius of a curve having the tangent 'T'



Let 'ϕ' be the angle between radius of a curve and tangent, let 'θ' be the angle between radius of a curve and initial line.

'ψ' be the angle made by the initial line.

WKT,

$$\Rightarrow \psi = \phi + \theta \quad \text{—— ①}$$

diff w.r to 'S'

$$\Rightarrow \frac{d\psi}{ds} = \frac{d\phi}{ds} + \frac{d\theta}{ds}$$

$$\Rightarrow \frac{d\psi}{ds} = \frac{d\phi/d\theta}{ds/d\theta} + \frac{d\theta}{ds}$$

$$\Rightarrow \frac{d\psi}{ds} = \frac{d\phi}{d\theta} \cdot \frac{d\theta}{ds} + \frac{d\theta}{ds}$$

$$\Rightarrow \frac{d\psi}{ds} = \frac{d\theta}{ds}\left[\frac{d\phi}{d\theta} + 1\right] \quad \text{—— ②}$$

Also, WKT

$$\Rightarrow \operatorname{Tan}\phi = r\frac{d\theta}{dr}$$

$$\Rightarrow \operatorname{Tan}\phi = \frac{r}{(dr/d\theta)}$$

$$\Rightarrow \operatorname{Tan}\phi = \frac{r}{r_1} \quad \text{where, } r_1 = \frac{dr}{d\theta}$$

$$\Rightarrow \phi = \tan^{-1}\left[r/r_1\right] \quad \text{—— ③}$$

using diff ③ w.r to 'θ'

$$\Rightarrow \frac{d\phi}{d\theta} = \frac{1}{1+\left[\frac{r}{r_1}\right]^2}\frac{d}{d\theta}\left[\frac{r}{r_1}\right]$$

$$\Rightarrow \frac{d\phi}{d\theta} = \left[\frac{1}{1+\frac{r^2}{r_1^2}}\right]\frac{r_1\frac{d}{d\theta}(r) - r\frac{d}{d\theta}(r_1)}{r_1^2}$$

$$\Rightarrow \frac{d\phi}{d\theta} = \frac{\cancel{r_1^2}}{r_1^2+r^2}\left[\frac{r_1r_1 - rr_2}{\cancel{r_1^2}}\right]$$

$$\Rightarrow \frac{d\phi}{d\theta} = \frac{r_1^2 - rr_2}{r_1^2+r^2}$$

$$\therefore ② \Rightarrow \frac{d\psi}{ds} = \frac{d\theta}{ds}\left[\frac{r_1^2 - rr_2}{r_1^2+r^2} + 1\right]$$

$$\Rightarrow \frac{1}{ds/d\psi} = \frac{1}{ds/d\theta}\left[\frac{r_1^2 - rr_2 + r_1^2 + r^2}{r_1^2+r^2}\right]$$

$$\Rightarrow \frac{1}{\rho} = \frac{1}{\sqrt{r^2+r_1^2}}\left[\frac{r^2+2r_1^2 - rr_2}{r^2+r_1^2}\right]$$

$$\Rightarrow \frac{1}{\rho} = \frac{r^2+2r_1^2-rr_2}{(r^2+r_1^2)^{3/2}}$$

$$\Rightarrow \rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}$$

RADIUS OF CURVATURE IN PARAMETRIC FORM :-

Let $x = x(t)$, $y = y(t)$ with a two parametric equations, then the radius of curvature can be evaluated from the given formula.

$$\Rightarrow \quad \rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - x''y'}$$

where, $\quad x' = \frac{dx}{dt} \quad , \quad y' = \frac{dy}{dt}$

$\quad x'' = \frac{d^2x}{dt^2} \quad , \quad y'' = \frac{d^2y}{dt^2}$

The Radius of Curvature of a polar curve in pedal form ;

$$\rho = r \frac{dr}{dp}$$

Problems :-

1. Find the radius of curvature of :-

$y = a \log(\sec x/a)$

$\Rightarrow$ diff w.r to o 'x'

$$\Rightarrow \frac{dy}{dx} = \frac{\cancel{a}}{\cancel{\sec\left[\frac{x}{a}\right]}} \cdot \tan\left[\frac{x}{a}\right] \cancel{\sec\left[\frac{x}{a}\right]}\left[\frac{1}{\cancel{a}}\right]$$

$$\Rightarrow \frac{dy}{dx} = \tan\left[\frac{x}{a}\right]$$

diff w.r to 'x'

$$\Rightarrow \frac{d^2y}{dx^2} = \sec^2\left(\frac{x}{a}\right)\left(\frac{1}{a}\right)$$

$$\Rightarrow y_2 = \frac{1}{a} \sec^2\left[\frac{x}{a}\right]$$

WKT

$$\Rightarrow \rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$\Rightarrow \rho = \frac{\left[1+\tan^2\left[\frac{x}{a}\right]\right]^{3/2}}{\frac{1}{a}\sec^2\left[\frac{x}{a}\right]}$$

$$\Rightarrow \rho = \frac{a\left[\sec^2\left(\frac{x}{a}\right)\right]^{3/2}}{\sec^2\left[\frac{x}{a}\right]}$$

$$\Rightarrow \rho = \frac{a\sec^3\left(\frac{x}{a}\right)}{\sec^2\left(\frac{x}{a}\right)}$$

$$\Rightarrow \rho = a\sec\left[\frac{x}{a}\right]$$

2. Find the radius of curvature at any point of the parabola $y^2 = 4ax$

$\Rightarrow \quad y^2 = 4ax$ —— ①

diff w.r to 'x'

① $\Rightarrow 2y \cdot \frac{dy}{dx} = 4a \cdot 1 \quad (\div 2)$

$\Rightarrow y\frac{dy}{dx} = 2a$

$\Rightarrow \frac{dy}{dx} = y_1 = \frac{2a}{y}$ —— ②

diff w.r to 'x'

② $\Rightarrow \frac{d}{dx}\left[\frac{dy}{dx}\right] = 2a \cdot \frac{d}{dx}\left[\frac{1}{y}\right]$

$\Rightarrow \frac{d^2y}{dx^2} = 2a \cdot \left[-\frac{1}{y^2}\right]\frac{dy}{dx}$

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{2a}{y^2}\left[\frac{2a}{y}\right]$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{4a^2}{y^3} = y_2$$

WKT,

$$\Rightarrow \rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$\Rightarrow \rho = \frac{\left[1+\left(\frac{2a}{y}\right)^2\right]^{3/2}}{-\frac{4a^2}{y^3}}$$

$$\Rightarrow \rho = \frac{\left[1+\frac{4a^2}{y^2}\right]^{3/2} \times y^3}{-4a^2}$$

$$\Rightarrow \rho = \frac{\frac{(y^2+4a^2)^{3/2}}{y^3} \times y^3}{4a^2}$$

$$\Rightarrow \rho = \frac{(y^2+4a^2)^{3/2}}{4a^2}$$

3. Find the radius of curvature at any point of the curve, $y = c\cosh(x/c)$

$$\Rightarrow \quad y = c\cosh\left[x/c\right] \text{ —— } ①$$

diff w.r to x

$$\Rightarrow ① \quad \frac{dy}{dx} = \cancel{c}\sinh\left[\frac{x}{c}\right]\left[\frac{1}{\cancel{c}}\right]$$

$$\Rightarrow \quad y_1 = \sinh\left[\frac{x}{c}\right] \text{ —— } ②$$

diff w.r.to x

② $\Rightarrow \frac{d}{dx}(y_1) = \cosh\left[\frac{x}{c}\right] \cdot \frac{1}{c}$

$\Rightarrow \quad y_2 = \frac{1}{c} \cosh\left[\frac{x}{c}\right]$

WKT,

$$\Rightarrow \quad \rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$\Rightarrow \quad \rho = \frac{\left[1+\sinh^2(x/c)\right]^{3/2}}{\frac{1}{c}\cos(x/c)}$$

$$\Rightarrow \quad \rho = \frac{c\left[\cosh^2[x/c]\right]^{3/2}}{\cosh(x/c)}$$

$$\Rightarrow \quad \rho = \frac{c\,\cosh^3(x/c)}{\cosh(x/c)}$$

$$\Rightarrow \quad \rho = c\,\cos^2h(x/c)$$

$$\Rightarrow \quad \rho = c\,(y/c)^2$$

$$\Rightarrow \quad \rho = \cancel{c} \cdot \frac{y^2}{c^{\cancel{2}}}$$

$$\Rightarrow \quad \rho = \frac{y^2}{c}$$

4. Find the radius of curvature of $x^3+y^3 = 3axy$ at the point $\left[\frac{3a}{2}, \frac{3a}{2}\right]$

$\Rightarrow \quad x^3+y^3 = 3axy$ —— ①

Let, $P = \left[\frac{3a}{2}, \frac{3a}{2}\right]$

diff w.r.to 'x'

$\Rightarrow \quad 3x^2 + 3y^2y_1 = 3a\left[1\cdot y + xy_1\right] \quad (\div 3)$

$$\Rightarrow x^2 + y^2 y_1 = ay + axy_1$$

$$\Rightarrow y^2 y_1 - axy_1 = ay - x^2$$

$$\Rightarrow y_1(y^2 - ax) = ay - x^2$$

$$\Rightarrow y_1 = \frac{ay - x^2}{y^2 - ax} \text{ ——— } ②$$

$$\Rightarrow (y_1)p = \frac{a\left(3a/2\right) - \left(\frac{3a}{2}\right)^2}{\left(\frac{3a}{2}\right)^2 - a\left[\frac{3a}{2}\right]}$$

$$\Rightarrow (y_1)p = \frac{\left[\frac{3a}{2}\right]^2 - a\left[\frac{3a}{2}\right]}{\left[\frac{3a}{2}\right]^2 - a\left[\frac{3a}{2}\right]}$$

$$\Rightarrow (y_1)p = -1$$

diff w.r to 'x'

$$② \Rightarrow y_2 = \frac{(y^2 - ax)(ay_1 - 2x) - (ay - x^2)(2yy_1 - a)}{y^2 - ax^2}$$

$$\Rightarrow y_2 p = \frac{\left[\left(\frac{3a}{2}\right)^2 - a\left(\frac{3a}{2}\right)\right]\left[a(-1) - 2\left[\frac{3a}{2}\right] - \left[a\left(\frac{3a}{2} - \frac{3a}{2}\right)^2\right]\right.\left[2\left[\frac{3a}{2}(-1) - a\right]\right.}{\left[\left(\frac{3a}{2}\right)^2 - 4\left(\frac{3a}{2}\right)\right]^2}$$

$$\Rightarrow (y_2)p = \frac{\left[\frac{9a^2}{4} - \frac{3a^2}{2}\right][-4a] - \left[-\frac{3a^2}{2} + \frac{9a^2}{4}\right][4a]}{\left[\frac{9a^2}{4} - \frac{3a^2}{2}\right]^2}$$

$$\Rightarrow (y_2)p = \frac{\left[\frac{9a^2}{4} - \frac{3a^2}{4}\right][-4a - 4a]}{\left[\frac{9a^2}{4} - \frac{3a^2}{4}\right]^2}$$

$$\Rightarrow y_2 = \frac{-8a}{\frac{9a^2-6a^2}{4}}$$

$$\Rightarrow y^2 = \left|\frac{8a\times 4}{3a^2}\right|$$

$$\Rightarrow y^2 = \frac{32}{3a}$$

WKT,

$$\Rightarrow \rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$\Rightarrow \rho = \frac{[1+(-1)^2]^{3/2}}{32/3a}$$

$$\Rightarrow \rho = \frac{2^{3/2}\times 3a}{3a}$$

$$\Rightarrow \rho = \frac{\cancel{2\sqrt{2}}\times 3a}{8\times \cancel{2\sqrt{2}}\times\sqrt{2}}$$

$$\Rightarrow \rho = \frac{3a}{8\sqrt{2}}$$

5. Find the radius of the curvature of a curve $x^2y = a(x^2+y^2)$ at the point $P(-2a, 2a)$

$$\Rightarrow x^2y = a(x^2+y^2) \quad \text{——— (1)}$$

$\Rightarrow$ where, $P = (-2a, 2a)$

diff w.r to 'x'

$$(1) \Rightarrow x^2(1) + y\,2xx_1 = a(2x\cdot x_1 + 2y)$$

$$\Rightarrow x^2 + 2xy\cdot x_1 = 2ax\cdot x_1 + 2ay$$

$$\Rightarrow 2xyx_1 - 2axx_1 = 2ay - x^2$$

$$\Rightarrow x_1(2xy - 2ax) = 2ay - x^2$$

$$\Rightarrow x_1 = \frac{2ay - x^2}{2xy - 2ax} \quad \text{——— (2)}$$

$$\therefore (x_1)_P = \frac{2a(2a) - (-2a^2)}{2(-2a)(2a) - 2a(-2a)}$$

$$(x_1)_P = \frac{4a^2 - 4a^2}{-8a^2 + 4a^2}$$

$$(x_1)_P = \frac{0}{-4a^2}$$

$$\Rightarrow (x_1)_P = 0$$

diff w.r to 'y'

$$(2) \Rightarrow x_2 = \frac{(2xy - 2ax)(2a - 2xx_1) - (2ay - x^2)(2x + 2xy - 2ax_1)}{(2xy - 2ax)^2}$$

$$\Rightarrow (x_2)_P = \frac{[2(-2a)(2a) - 2a(-2a)][2a - 2(-2a)(0)] - [2a(2a) - (-2a)^2][2[-2a] - 2(0) - 0]}{[2[-2a][2a] - 2a[-2a]]^2}$$

$$\Rightarrow (x_2)_P = \frac{(-8a^2 + 4a^2) - 0}{(-8a^2 + 4a^2)^2}$$

$$\Rightarrow (x_2)_P = \frac{(2a) - (4a^2)}{(-4a^2)^2}$$

$$\Rightarrow x_2 = \frac{2a}{-4a^2} = \frac{1}{-2a}$$

$$\Rightarrow \rho = \frac{(1 + x_1^2)^{3/2}}{x_2} = \frac{(1 + (0)^2)^{3/2}}{1/-2a} = \frac{(1+0)^{3/2}}{1/-2a} = \frac{(1)^{3/2} \times (-2a)}{1}$$

$\Rightarrow \rho = \dfrac{\cancel{1}\sqrt{2} \times (-3a)}{\cancel{1}}$

$\Rightarrow \rho = \sqrt{2}(-3a)$

6. Find the radius of a curvature the curve $a^2y = x^3 - a^3$ at the point where the curve cuts 'x' axis.

$\Rightarrow$ Given:-

$$a^2y = x^3 - a^3 \quad \text{——— (1)}$$

Let, p be a point cuts the x-axis $(y=0)$

(1) $\Rightarrow \quad 0 = x^3 - a^3$

$\Rightarrow x^3 = a^3$

$\Rightarrow x = a$

$\therefore p(a, 0)$

diff (1) w.r to 'x'

(1) $\Rightarrow a^2y_1 = 3x^2 - 0$

$\Rightarrow y_1 = \dfrac{3x^2}{a^2} \quad \text{——— (2)}$

$\Rightarrow (y_1)p = \dfrac{3\cancel{a^2}}{\cancel{a^2}}$

$\Rightarrow (y_1) = 3$

$\Rightarrow y_2 = \dfrac{3}{a^2}(2x)$

$\Rightarrow y_2 = \dfrac{6x}{a^2}$

$\Rightarrow (y_2)p = \dfrac{6\cancel{a}}{a^{\cancel{2}}}$

$\Rightarrow y_2 = \dfrac{6}{a}$

$$\therefore \rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$\Rightarrow \rho = \frac{(1+9)^{3/2}}{6/a}$$

$$\Rightarrow \rho = \frac{a(10)^{3/2}}{6}$$

$$\Rightarrow \rho = \frac{a \times \cancel{10}^{5} \times \sqrt{10}}{\cancel{6}\,3}$$

$$\Rightarrow \rho = \frac{5\sqrt{10}\,.\,a}{3}$$

7. For the curve $y = \frac{ax}{a+x}$, prove that

$$\left[\frac{2\rho}{a}\right]^{2/3} = \left[\frac{y}{x}\right]^2 + \left[\frac{x}{y}\right]^2,$$ where, ρ is the radius of curvature at its point $p(x, y)$

$$\Rightarrow \quad y = \frac{ax}{a+x} \quad \text{——— (1)}$$

diff w.r to 'x'

$$\Rightarrow \quad y_1 = \frac{(a+x)(a) - ax(1)}{(a+x)^2}$$

$$\Rightarrow \quad y_1 = \frac{a^2 + \cancel{ax} - \cancel{ax}}{(a+x)^2}$$

$$\Rightarrow \quad y_1 = \frac{a^2}{(a+x)^2} \quad \text{——— (2)}$$

diff w.r to 'x'

$$\Rightarrow \quad y_2 = \frac{(a+x)^2(0) - a^2 \cdot 2(a+x)(1)}{(a+x)^4}$$

$$\Rightarrow \quad y_2 = \frac{-2a^2(a+x)}{(a+x)^4}$$

$$\Rightarrow \quad y_2 = \frac{-2a^2}{(a+x)^3}$$

∴ WKT;

$$\Rightarrow \rho = \left| \frac{(1+y_1^2)^{3/2}}{y_2} \right|$$

$$\Rightarrow \rho = \left| \frac{\left[1 + \frac{a^4}{(a+x)^4}\right]^{3/2}}{\frac{-2a^2}{(a+x)^3}} \right|$$

$$\Rightarrow \rho = \frac{(a+x)^3}{2a^2}\left[1 + \frac{a^4}{(a+x)^4}\right]^{3/2}$$

$$\Rightarrow \frac{2\rho}{a} = \frac{\cancel{2}}{a}\,\frac{(a+x)^3}{\cancel{2}a^2}\left[1 + \frac{a^4}{(a+x)^4}\right]^{3/2}$$

$$\Rightarrow \frac{2\rho}{a} = \frac{(a+x)^3}{a^3}\left[1 + \frac{a^4}{(a+x)^4}\right]^{3/2}$$

$$\Rightarrow \left[\frac{2\rho}{a}\right]^{2/3} = \frac{[(a+x)^3]^{2/3}}{(a^3)^{2/3}}\left[1 + \frac{a^4}{(a+x)^4}\right]^{\cancel{3}/\cancel{2} \times \cancel{2}/\cancel{3}}$$

$$\Rightarrow \left[\frac{2\rho}{a}\right]^{2/3} = \frac{(a+x)^2}{a^2}\left[1 + \frac{a^4}{(a+x)^4}\right]$$

$$\Rightarrow \left[\frac{2\rho}{a}\right]^{2/3} = \frac{(a+x)^2}{a^2} + \frac{a^2}{(a+x)^2}$$

$$\Rightarrow \left[\frac{2\rho}{a}\right]^{2/3} = \left(\frac{a+x}{a}\right)^2 + \left[\frac{a}{a+x}\right]^2$$

$$\Rightarrow \left[\frac{2\rho}{a}\right]^{2/3} = \left[\frac{x}{y}\right]^2 + \left[\frac{y}{x}\right]^2$$

8. Find the radius of curvature at any point of the cycloid.

$x = a(\theta + \sin\theta) \qquad y = a(1 - \cos\theta)$

$\Rightarrow x = a(\theta + \sin\theta)$ —— ① $\qquad \Rightarrow y = a(1 - \cos\theta)$ —— ②

diff w.r to 'θ' $\qquad$ diff w.r to θ

$$\Rightarrow \frac{dx}{d\theta} = a[1 + \cos\theta] \qquad \Rightarrow \frac{dy}{d\theta} = a[0 + \sin\theta]$$

$$\Rightarrow \frac{dy}{d\theta} = a\sin\theta$$

$$\therefore y_1 = \frac{dy/d\theta}{dx/d\theta}$$

$$\Rightarrow y_1 = \frac{\cancel{a}\sin\theta}{\cancel{a}(1+\cos\theta)} = \frac{\sin\theta}{1+\cos\theta} \text{ —— ③}$$

diff eqn ③ w.r to x

$$③ \Rightarrow \frac{d}{dx}(y_1) = \frac{d}{dx}\left[\frac{\sin\theta}{1+\cos\theta}\right]$$

$$\Rightarrow \frac{d}{dx}(y_1) = \frac{d}{d\theta}\left[\frac{\sin\theta}{1+\cos\theta}\right]\frac{d\theta}{dx}$$

$$= \frac{(1+\cos\theta)\cos\theta - \sin\theta(-\sin\theta)}{(1+\cos\theta)^2}\,\frac{d\theta}{dx}$$

$$= \frac{\cos\theta + (\cos^2\theta + \sin^2\theta)}{(1+\cos\theta)^2}\,\frac{1}{dx/d\theta}$$

$$\frac{1+\cos\theta}{(1+\cos\theta)^2} \times \frac{1}{a(1+\cos\theta)}$$

$$\Rightarrow y_2 = \frac{1}{a(1+\cos\theta)^2}$$

WKT,

$$\Rightarrow \rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$\Rightarrow \rho = \frac{\left[1+\frac{\sin^2\theta}{(1+\cos\theta)^2}\right]^{3/2}}{\frac{1}{a+[\cos\theta+1]^2}}$$

$$\Rightarrow \rho = a(1+\cos\theta)^2\left[\frac{[(1+\cos\theta)^2+\sin^2\theta]^{3/2}}{[(1+\cos\theta)^2]^{3/2}}\right]$$

$$\Rightarrow \rho = \frac{a(1+\cos\theta)^2}{(1+\cos\theta)^3}\left[1+2\cos\theta+\cos^2\theta+\sin^2\theta\right]^{3/2}$$

$$\Rightarrow \rho = \frac{a}{1+\cos\theta}\left[1+2\cos\theta+1\right]^{3/2}$$

$$\Rightarrow \rho = \frac{a}{1+\cos\theta}\left[2+2\cos\theta\right]^{3/2}$$

$$\Rightarrow \rho = \frac{2^{3/2}a}{(1+\cos\theta)}(1+\cos\theta)^{3/2}$$

9. Find the radius of curvature of curve $\sqrt{x}+\sqrt{y}=\sqrt{a}$ at the point $\left[\frac{a}{4},\frac{a}{4}\right]$

$$\Rightarrow \sqrt{x}+\sqrt{y}=\sqrt{a} \quad \text{——} (1) \qquad P\left[\frac{a}{4},\frac{a}{4}\right]$$

diff w.r to x

$$(1) \Rightarrow \frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\frac{dy}{dx}=0$$

$$\Rightarrow \frac{1}{\sqrt{y}} \frac{dy}{dx} = -\frac{1}{\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}} = y_1 \quad \text{——— (2)}$$

$$\therefore (y_1)p = \frac{-\sqrt{a/4}}{\sqrt{a/4}} = -1$$

diff w.r to x

$$\Rightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\sqrt{x}\cdot\frac{1}{2\sqrt{y}}\cdot y_1 - \sqrt{y}\cdot\frac{1}{2\sqrt{x}}}{(\sqrt{x})^2}$$

$$\Rightarrow y_2 = -\frac{1}{2}\,\frac{y_1\sqrt{x}/\sqrt{y} - \sqrt{y}/\sqrt{x}}{x}$$

$$\Rightarrow (y_2)p = -\frac{1}{2}\left[\frac{\frac{(-1)\sqrt{a/4} - \sqrt{a/4}}{\sqrt{a/4} - \sqrt{a/4}}}{a/4}\right]$$

$$\Rightarrow (y_2)p = -\frac{1}{2}\left[\frac{-1-1}{a/4}\right]$$

$$\Rightarrow y_2 = -\frac{1}{\cancel{2}}\left[\frac{-\cancel{2}}{a/4}\right]$$

$$\Rightarrow y_2 = \frac{4}{a} \neq 0$$

$\therefore$ WKT,

$$\Rightarrow \rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$\Rightarrow \rho = \frac{[1+(-1)^2]^{3/2}}{4/a}$$

$$\Rightarrow \rho = \frac{\cancel{2}\sqrt{2}}{{}_2\cancel{4}/a}$$

$$\Rightarrow \rho = \frac{\sqrt{2}}{2} a$$

$$\Rightarrow \rho = \frac{\sqrt{\cancel{2}}\, a}{\sqrt{\cancel{2}}\sqrt{2}}$$

$$\Rightarrow \rho = \frac{a}{\sqrt{2}}$$

10. Find the radius of curvature for the curve $r^n = a^n \cos n\theta$

$\Rightarrow$ Given:-

$$\Rightarrow r^n = a^n \cos n\theta \quad \text{——— (1)}$$

diff w.r.to 'θ'

$$(1) \Rightarrow \cancel{n} r^{n-1} \frac{dr}{d\theta} = a^n [-\sin n\theta][\cancel{n}]$$

$$\Rightarrow \frac{r^n}{r} \frac{dr}{d\theta} = -a^n \sin n\theta$$

$$\Rightarrow \frac{1}{r} \cdot \frac{dr}{d\theta} = -\frac{a^n \sin n\theta}{r^n}$$

$$\Rightarrow \frac{1}{r} \cdot \frac{dr}{d\theta} = -\frac{\cancel{a^n} \sin n\theta}{\cancel{a^n} \cos n\theta}$$

$$\Rightarrow \cot\phi = -\tan n\theta \quad \text{——— (2)}$$

$\therefore$ WKT

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}[1 + \cot^2\phi]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}[1 + \tan^2 n\theta]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}[\sec^2 n\theta]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2\cos^2 n\theta}$$

$$\Rightarrow p^2 = r^2\cos^2 n\theta$$

$$\Rightarrow p = r\cos n\theta$$

$$\Rightarrow p = r\,\frac{r^n}{a^n}$$

$$\Rightarrow a^n p = r^{n+1} \quad \text{——} ②$$

diff w.r to 'p'

$$② \Rightarrow a^n.(1) = (n+1)\,r^n \cdot \frac{dr}{dp}$$

$$\Rightarrow r^n \frac{dr}{dp} = \frac{a^n}{n+1}$$

$$\Rightarrow r^{n-1}\cdot r\,\frac{dr}{dp} = \frac{a^n}{n+1}$$

$$\Rightarrow r\,\frac{dr}{dp} = \frac{a^n}{n+1}\cdot\frac{1}{r^{n-1}}$$

$$\Rightarrow \rho = \left[\frac{a^n}{n+1}\right]\frac{1}{r^{n-1}}$$

$$\Rightarrow \rho \propto \frac{1}{r^{n-1}}$$

11. $r^n = a^n \sin n\theta$

$$\Rightarrow r^n = a^n \sin n\theta \quad \text{——} ①$$

diff w.r to 'θ'

$$\Rightarrow \cancel{n}\,r^{n-1}\frac{dr}{d\theta} = a^n(\cos n\theta)(\cancel{n})$$

$$\Rightarrow \frac{r^n}{r}\,\frac{dr}{d\theta} = a^n\cos n\theta$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{a^n \cos n\theta}{r^n}$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{\cancel{a^n}\cos n\theta}{\cancel{a^n}\sin n\theta}$$

$$\Rightarrow \cot\phi = \cot n\theta \quad \text{——— } (2)$$

WKT ;

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}[1+\cot^2\phi]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}[1+\cot^2 n\theta]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}(\operatorname{cosec}^2 n\theta)$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2\sin^2 n\theta}$$

$$\Rightarrow p^2 = r^2\sin^2 n\theta$$

$$\Rightarrow p = r\sin n\theta$$

$$\Rightarrow p = r\,\frac{r^n}{a^n}$$

$$\Rightarrow a^n p = r^{n+1} \quad \text{——— } (3)$$

diff w.r to 'p'

$$(3) \Rightarrow a^n.1 = (n+1)r^n.\frac{dr}{dp}$$

$$\Rightarrow r^n\frac{dr}{dp} = \frac{a^n}{n+1}$$

$$\Rightarrow \frac{r\,dr}{dp} = \frac{a^n}{n+1}\cdot\frac{1}{r^{n-1}}$$

$$\Rightarrow \rho = \left(\frac{a^n}{n+1}\right)\frac{1}{r^{n-1}} \qquad \Rightarrow \rho \propto \frac{1}{r^{n-1}}$$

12. Find $\frac{2a}{r} = 1+\cos\theta$ and hence show that $\rho^2 \propto r^3$ (or) $\frac{\rho^2}{r^3}$ be a constant.

$$\Rightarrow \quad \frac{2a}{r} = 1+\cos\theta$$

$$\Rightarrow \quad r(1+\cos\theta) = 2a \longrightarrow (1)$$

diff w.r to 'θ'

$$\Rightarrow (1+\cos\theta)\frac{dr}{d\theta} + r(0-\sin\theta) = 0$$

$$\Rightarrow (1+\cos\theta)\frac{dr}{d\theta} = r\sin\theta$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{\sin\theta}{1+\cos\theta}$$

$$\Rightarrow \cot\phi = \frac{\cancel{2}\sin\theta/2\,\cancel{\cos\theta/2}}{\cancel{2}\cos^{\cancel{2}}\theta/2}$$

$$\Rightarrow \cot\phi = \tan(\theta/2)$$

WKT;

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}(1+\cot^2\phi)$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\left(1+\tan^2\left(\frac{\theta}{2}\right)\right)$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}(\sec^2\theta/2)$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2\cos^2\theta/2}$$

$$\Rightarrow p^2 = r^2\cos^2\theta/2$$

$$\Rightarrow p^2 = r^2\left[\frac{1+\cos\theta}{2}\right]$$

$$\Rightarrow p^2 = \frac{r^2}{2}[1+\cos\theta]$$

$$\Rightarrow p^2 = \frac{r^2}{2}\left[\frac{2a}{r}\right]$$

$$\Rightarrow \quad p^2 = ar \quad \text{—— (2)}$$

diff w.r to 'p'

$$(2) \Rightarrow \quad 2p = a\frac{dr}{dp}$$

$$\Rightarrow \frac{dr}{dp} = \frac{2}{a}p$$

$$\Rightarrow r\frac{dr}{dp} = \frac{2}{a}pr$$

$$\Rightarrow \rho = \frac{2}{a}pr$$

$$\Rightarrow \rho^2 = \frac{4}{a^2}p^2r^2$$

$$\Rightarrow \rho^2 = \frac{4}{a^2}(ar)(r^2)$$

$$\Rightarrow \rho^2 = \frac{4}{a}r^3$$

$$\Rightarrow \rho^2 = Kr^3 \qquad \therefore K = 4/a$$

$$\Rightarrow \rho^2 \propto r^3$$

(or)

$$\Rightarrow \frac{\rho^2}{r^3} = K = \text{constant.}$$

13. $$\frac{2a}{r} = (1-\cos\theta)$$

$$\Rightarrow \quad 2a = r(1-\cos\theta) \quad \text{—— (1)}$$

diff w.r to 'θ'

$$\Rightarrow \quad 0 = (1-\cos\theta)\frac{dr}{d\theta} - (-r\sin\theta)$$

$$\Rightarrow -r\sin\theta = (1-\cos\theta)\frac{dr}{d\theta}$$

$$\Rightarrow \frac{-\sin\theta}{1-\cos\theta} = r\frac{d\theta}{dr}$$

$$\Rightarrow \frac{-\cancel{2}\sin\theta/2\cos\theta/2}{\cancel{2}\sin^2\theta/2} = \cot\phi$$

$$\Rightarrow -\cot\theta/2 = \cot\phi$$

$$\Rightarrow \cot\phi = \cot\left[-\frac{\theta}{2}\right]$$

WKT;

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\left[1+\cot^2\phi\right]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\left[1+\cot^2\left[-\frac{\theta}{2}\right]\right]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\left[\operatorname{cosec}^2\frac{\theta}{2}\right]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2\sin^2\theta/2}$$

$$\Rightarrow p^2 = r^2\sin^2\frac{\theta}{2}$$

$$\Rightarrow p^2 = r^2\sin^2\theta/2$$

$$\Rightarrow p^2 = r^2\left[\frac{1-\cos\theta}{2}\right]$$

$$\Rightarrow p^2 = \frac{r^2}{2}\left[1-\cos\theta\right]$$

$$\Rightarrow p^2 = \frac{r^{\cancel{2}}}{\cancel{2}}\left(\frac{2a}{r}\right)$$

$$\Rightarrow p^2 = ar \quad \text{——} (2)$$

$\Rightarrow 2p = r\dfrac{dr}{dp}$

$\Rightarrow \dfrac{dr}{dp} = \dfrac{2}{a}p$

$\Rightarrow r\dfrac{dr}{dp} = \dfrac{2}{a}pr$

$\Rightarrow \rho = \dfrac{2}{a}pr$

$\Rightarrow \rho^2 = \dfrac{4}{a^2}p^2r^2$

$\Rightarrow \rho^2 = \dfrac{4}{a^{\cancel{2}}}(\cancel{a}r)(r^2)$

$\Rightarrow \rho^2 = \dfrac{4}{a}r^3$

$\Rightarrow \rho^2 = Kr^3 \qquad \therefore K = 4/a$

$\Rightarrow \rho^2 \propto r^3$

(or)

$\dfrac{\rho^2}{r^3} = K = \text{constant.}$

14. $r = a(1-\cos\theta)$

$\Rightarrow r = a(1-\cos\theta)$ —— (1)

diff w.r to 'θ'

$\Rightarrow \dfrac{dr}{d\theta} = a\sin\theta$

$\Rightarrow \dfrac{1}{r}\dfrac{dr}{d\theta} = \dfrac{a\sin\theta}{r}$

$\Rightarrow \dfrac{1}{r}\dfrac{dr}{d\theta} = \dfrac{\cancel{a}\sin\theta}{\cancel{a}(1-\cos\theta)}$

$$\Rightarrow \cot\phi = \frac{\cancel{2}\cancel{\sin\theta/2}\cos\theta/2}{\cancel{2}\sin^2\theta/2}$$

$$\Rightarrow \cot\phi = \cot\frac{\theta}{2}$$

WKT,

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\left[1+\cot^2\phi\right]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\left[1+\cot^2\frac{\theta}{2}\right]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\operatorname{cosec}^2\frac{\theta}{2}$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2\sin^2\theta/2}$$

$$\Rightarrow p^2 = r^2\sin^2\theta/2$$

$$\Rightarrow p^2 = r^2\left[\frac{1-\cos\theta/2}{2}\right]$$

$$\Rightarrow p^2 = \frac{r^2}{2}\left[1-\cos\theta/2\right]$$

$$\Rightarrow p^2 = \frac{r^2}{2}\left[\frac{r}{a}\right]$$

$$\Rightarrow p^2 = \frac{r^3}{2a}$$

$$\Rightarrow 2ap^2 = r^3 \quad \text{—— (2)}$$

diff (2) w r to 'p'

$$(2) \Rightarrow 4ap = 3r^2 \cdot \frac{dr}{dp}$$

$$\Rightarrow r^2 \cdot \frac{dr}{dp} = \frac{4a}{3} p$$

$$\Rightarrow r \cdot \frac{dr}{dp} = \frac{4a}{3} \cdot \frac{p}{r}$$

$$\Rightarrow \rho = \frac{4a}{3} \cdot \frac{p}{r}$$

$$\Rightarrow \rho^2 = \frac{16a^2}{9} \cdot \frac{p^2}{r^2}$$

$$\Rightarrow \rho^2 = \frac{16a^2}{9} \cdot \frac{1}{r^2}\left[\frac{r^3}{2a}\right]$$

$$\Rightarrow \rho^2 = \frac{8a}{9} r$$

$$\Rightarrow \rho^2 \propto r$$

15. Find the radius of curvature of the curve

$$\theta = \frac{\sqrt{r^2-a^2}}{a} - \cos^{-1}\left[\frac{a}{r}\right]$$

$$\Rightarrow \theta = \frac{\sqrt{r^2-a^2}}{a} - \cos^{-1}\left[\frac{a}{r}\right] \longrightarrow (1)$$

diff w. r to 'r'

$$\Rightarrow \frac{d\theta}{dr} = \frac{1}{a}\frac{1}{2\sqrt{r^2-a^2}}(2r) - \frac{(-1)}{\sqrt{1-\left[\frac{a}{r}\right]^2}} \cdot \frac{d}{dr}\left[\frac{a}{r}\right]$$

$$\Rightarrow \frac{d\theta}{dr} = \frac{r}{a\sqrt{r^2-a^2}} + \frac{1}{\sqrt{1-\frac{a^2}{r^2}}}\left[\frac{-a}{r^2}\right]$$

$$\Rightarrow \frac{d\theta}{dr} = \frac{r}{a\sqrt{r^2-a^2}} - \frac{a}{\frac{\sqrt{r^2-a^2}}{r}}\left[\frac{1}{r^2}\right]$$

$$\Rightarrow \frac{d\theta}{dr} = \frac{r}{a\sqrt{r^2-a^2}} - \frac{a}{r\sqrt{r^2-a^2}}$$

$$\Rightarrow \frac{d\theta}{dr} = \frac{1}{\sqrt{r^2-a^2}}\left[\frac{r}{a}-\frac{a}{r}\right]$$

$$\Rightarrow \frac{d\theta}{dr} = \frac{1}{\sqrt{r^2-a^2}}\left[\frac{r^2-a^2}{ar}\right]$$

$$\Rightarrow \frac{d\theta}{dr} = \frac{1}{\sqrt{r^2-a^2}} \cdot \frac{\sqrt{r^2-a^2}\cdot\sqrt{r^2-a^2}}{ar}$$

$$\Rightarrow \frac{d\theta}{dr} = \frac{\sqrt{r^2-a^2}}{ar}$$

$$\Rightarrow \frac{dr}{d\theta} = \frac{ar}{\sqrt{r^2-a^2}}$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{a}{\sqrt{r^2-a^2}}$$

$$\Rightarrow \cot\phi = \frac{a}{\sqrt{r^2-a^2}}$$

$$\Rightarrow \cot^2\phi = \frac{a^2}{r^2-a^2}$$

$$\Rightarrow 1+\cot^2\phi = 1+\frac{a^2}{r^2-a^2}$$

$$\Rightarrow 1+\cot^2\phi = \frac{r^2-a^2+a^2}{r^2-a^2} = \frac{r^2}{r^2-a^2}$$

WKT,

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\left[1+\cot^2\phi\right]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\times\frac{r^2}{r^2-a^2}$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2-a^2}$$

$$\Rightarrow p^2 = r^2-a^2$$

$$\Rightarrow p = \sqrt{r^2-a^2} \quad \text{---}(2)$$

diff wr to 'p'

$$(2) \Rightarrow 1 = \frac{1}{2\sqrt{r^2-a^2}}\cdot 2r\cdot\frac{dr}{dp} \Rightarrow r\frac{dr}{dp} = \sqrt{r^2-a^2}$$

$$\Rightarrow \rho = \sqrt{r^2-a^2}$$