

Series Expansion:-

Let $y = f(x)$ be a real valued function defined in the (a, b) and it can be expressed as a series given below:

$$1. f(b) = f(a) + \frac{(b-a)^1}{1!} f'(a) + \frac{(b-a)^2}{2!} f''(a) + \frac{(b-a)^3}{3!} f'''(a) + \dots \quad (1)$$

$$\text{Let, } b-a=h \\ \Rightarrow b=a+h$$

$$\Rightarrow f(a+h) = f(a) + \frac{h^1}{1!} f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \dots \quad (2)$$

$$\text{Let, } a+h=x \\ \Rightarrow x=h+a \\ \Rightarrow h=x-a$$

$$\therefore \Rightarrow f(x) = f(a) + \frac{x-a}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \dots \quad (3)$$

$$y(x) = y(a) + \frac{(x-a)^1}{1!} y'(a) + \frac{(x-a)^2}{2!} y''(a) + \frac{(x-a)^3}{3!} y'''(a) + \dots \quad (4)$$

Therefore, eqn (3) and (4) are called the Taylor's series expansion of $y = f(x)$ about $x=a$ (or) in the powers of $(x-a)$.

2. When, $(a=0)$ the series eqn (3) and (4) can be expressed as,

$$\textcircled{3} \Rightarrow f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots$$

$$\textcircled{4} \Rightarrow y(x) = y(0) + \frac{x}{1!} y'(0) + \frac{x^2}{2!} y''(0) + \frac{x^3}{3!} y'''(0) + \dots \quad (\text{OR})$$

Which is called the "Maclaurin's series of Expansion".

Problems:-

1. Find Taylor's series expansion of $\sin(x)$ in the powers of $(x - \pi/2)$

$$\Rightarrow \text{Let, } y = \sin x \text{ and } a = \pi/2$$

$$\therefore y(a) = y(\pi/2) \Rightarrow \sin(\pi/2) = 1$$

$$y'(a) = \cos x \Rightarrow y'(a) = y'(\pi/2) \Rightarrow \cos(\pi/2) = 0$$

$$y''(a) = -\sin x \Rightarrow y''(a) = y''(\pi/2) \Rightarrow -\sin(\pi/2) = -1$$

$$y'''(a) = -\cos x \Rightarrow y'''(a) = y'''(\pi/2) \Rightarrow -\cos(\pi/2) = 0$$

$$y^{IV}(a) = \sin x \Rightarrow y^{IV}(a) = y^{IV}(\pi/2) \Rightarrow \sin(\pi/2) = 1$$

$\therefore$ WKT,

$$y(x) = y(a) + \frac{(x-a)}{1!} y'(a) + \frac{(x-a)^2}{2!} y''(a) + \frac{(x-a)^3}{3!} y'''(a) + \dots$$

$$\Rightarrow y(x) = 1 + \frac{(x - \pi/2)}{1!} (0) + \frac{(x - \pi/2)^2}{2!} (-1) + \frac{(x - \pi/2)^3}{3!} (0) + \frac{(x - \pi/2)^4}{4!} (1) - \dots$$

$$\Rightarrow \sin x = 1 - \frac{(x - \pi/2)^2}{2!} + \frac{(x - \pi/2)^4}{4!} + \dots$$

2. $\log x$

$$\Rightarrow \therefore \text{Let } y = \log x, \quad a = 1$$

$$y(a) = y(1) = \log(1) = 0$$

$$y'(a) = \frac{1}{x} \Rightarrow y'(a) = y'(1) = \frac{1}{1} = 1$$

$$y''(a) = -\frac{1}{x^2} \Rightarrow y''(a) = y''(1) = -\frac{1}{1^2} = -1$$

$$y'''(a) = \frac{2}{x^3} \Rightarrow y'''(a) = y'''(1) = \frac{2}{1^3} = 2$$

$$y^{(4)}(a) = -\frac{6}{x^4} \Rightarrow y^{(4)}(a) = y^{(4)}(1) = -\frac{6}{1^4} = -6$$

$\therefore$ WKT,

$$\Rightarrow y(x) = y(a) + \frac{(x-a)}{1!} y'(a) + \frac{(x-a)^2}{2!} y''(a) + \frac{(x-a)^3}{3!} y'''(a) + \frac{(x-a)^4}{4!} y^{(4)}(a) + \dots$$

$$\Rightarrow y(x) = 0 + \frac{(x-1)}{1!} (1) + \frac{(x-1)^2}{2!} (-1) + \frac{(x-1)^3}{3!} (2) + \frac{(x-1)^4}{4!} (-6) + \dots$$

$$\Rightarrow \log x = \frac{(x-1)}{1!} - \frac{(x-1)^2}{2!} + \frac{2(x-1)^3}{3!} - \frac{6(x-1)^4}{4!} + \dots$$

$$\Rightarrow \log x = \frac{(x-1)}{1!} - \frac{(x-1)^2}{2!} + \frac{2(x-1)^3}{6} - \frac{6(x-1)^4}{24} + \dots$$

$$\Rightarrow \log x = \frac{(x-1)}{1} - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \dots$$

3. Expand the function e^x using Maclaurin's series.

$\Rightarrow$ Let, $y = e^x$

$$\therefore y(x) = e^x \Rightarrow y(0) = e^0 = 1$$

$$y'(x) = e^x \Rightarrow y'(0) = e^0 = 1$$

$$y''(x) = e^x \Rightarrow y''(0) = e^0 = 1$$

$\therefore$ WKT,

$$y(x) = y(0) + \frac{x}{1!} y'(0) + \frac{x^2}{2!} y''(0) + \frac{x^3}{3!} y'''(0) + \frac{x^4}{4!} y^{(4)}(0) + \dots$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

4. Expand the function $\sqrt{1 + \sin 2x}$ using Maclaurin's series;

$\Rightarrow$ Let, $y = \sqrt{1 + \sin 2x}$

$$\Rightarrow y = \sqrt{\sin^2 x + \cos^2 x + 2 \sin x \cos x}$$

$$\Rightarrow y = \sqrt{(\sin x + \cos x)^2}$$

$$\Rightarrow y = \sin x + \cos x$$

$$\therefore y(0) = \sin 0 + \cos 0 \Rightarrow 0 + 1 = 1$$

$$y'(x) = \cos x - \sin x \Rightarrow y'(0) = \cos(0) - \sin(0) = 1 - 0 = 1$$

$$y''(x) = -\sin x - \cos x \Rightarrow y''(0) = -\sin(0) - \cos(0) = 0 - 1 = -1$$

$$y'''(x) = -\cos x + \sin x \Rightarrow y'''(0) = -\cos(0) + \sin(0) = -1 + 0 = -1$$

WKT,

$$y(x) = y(0) + \frac{x}{1!} y'(0) + \frac{x^2}{2!} y''(0) + \frac{x^3}{3!} y'''(0) + \frac{x^4}{4!} y^{(4)}(0) + \dots$$

$$\Rightarrow \sqrt{1 + \sin 2x} = 1 + \frac{x}{1!} (1) + \frac{x^2}{2!} (-1) + \frac{x^3}{3!} (-1) + \dots$$

$$\Rightarrow \sqrt{1+\sin 2x} = 1 + \frac{x}{1!} - \frac{x^2}{2!} - \frac{x^3}{3!} - \dots$$

5. Expand the function $y = \log(1+\sin x)$ using Maclaurin's series upto the term containing x^4 .

$$\Rightarrow \text{Let, } y = \log(1+\sin x)$$

$$\Rightarrow y(x) = \log(1+\sin x) = y(0) = \log(1+\sin 0) = \log 1 = 0$$

$$\Rightarrow y'(x) = \frac{\cos x}{1+\sin x} \Rightarrow y'(0) = \frac{\cos 0}{1+\sin 0} = \frac{1}{1+0} = 1$$

$$\Rightarrow y''(x) = \frac{(1+\sin x)(-\sin x) - \cos x(\cos x)}{(1+\sin x)^2}$$

$$= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2}$$

$$= \frac{-\sin x - (\sin^2 x + \cos^2 x)}{(1+\sin x)^2}$$

$$= \frac{-\sin x - 1}{(1+\sin x)^2}$$

$$= \frac{-(1+\sin x)}{(1+\sin x)^2}$$

$$= \frac{-1}{(1+\sin x)}$$

$$= \frac{-1}{1+\sin(0)} = \frac{-1}{1+0} = -1$$

$$\Rightarrow y'''(x) = - \left[\frac{1+\sin x(0) - 1(\cos x)}{(1+\sin x)^2} \right]$$

$$y'''(x) = \frac{\cos x}{(1+\sin x)^2} \Rightarrow y'''(0) = \frac{\cos 0}{(1+\sin 0)^2} = \frac{1}{(1+0)^2} = 1$$

WKT,

$$y(x) = y(0) + \frac{x}{1!} y'(0) + \frac{x^2}{2!} y''(0) + \frac{x^3}{3!} y'''(0) + \dots$$

$$\Rightarrow \log(1 + \sin x) = 1 + \frac{x}{1!} (1) + \frac{x^2}{2!} (-1) + \frac{x^3}{3!} (1) + \frac{x^4}{4!} (-1) + \dots$$

$$\Rightarrow \log(1 + \sin x) = 1 + \frac{x}{1!} - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \dots$$

6. Expand $y = \log(1 + \cos x)$ using Maclaurin's expansion.

$$\Rightarrow \text{Let, } y = \log(1 + \cos x)$$

$$\therefore y(x) = \log(1 + \cos x) \Rightarrow y(0) = \log(1 + \cos 0) = \log(1 + 1) = \log 2$$

$$\Rightarrow y'(x) = \frac{-\sin x}{1 + \cos x} = \frac{-\sin 0}{1 + \cos 0} = \frac{-0}{1+1} = 0$$

$$\Rightarrow y''(x) = \frac{(1 + \cos x)(-\cos x) - \sin x(-\sin x)}{(1 + \cos x)^2}$$

$$= \frac{-\cos x - \cos^2 x + \sin^2 x}{(1 + \cos x)^2}$$

$$= \frac{-\cos x - (\sin^2 x + \cos^2 x)}{(1 + \cos x)^2}$$

$$= \frac{-\cos x - 1}{(1 + \cos x)^2}$$

$$= \frac{-(1 + \cos x)}{(1 + \cos x)^2}$$

$$= \frac{-1}{1 + \cos x} = \frac{-1}{1 + \cos 0} = \frac{-1}{1+1} = -\frac{1}{2}$$

$$\begin{aligned}
 \Rightarrow y'''(x) &= - \left[\frac{(1+\cos x)(0) - 1(-\sin x)}{(1+\cos x)^2} \right] \\
 &= \frac{\sin x}{(1+\cos x)^2} \\
 &= \frac{\sin 0}{(1+\cos 0)^2} = \frac{0}{(1+1)^2} = \frac{0}{4} = 0
 \end{aligned}$$

$\therefore$ WKT,

$$\Rightarrow y(x) = y(0) + \frac{x}{1!} y'(0) + \frac{x^2}{2!} y''(0) + \frac{x^3}{3!} y'''(0) + \dots$$

$$\Rightarrow \log(1+\cos x) = \log 2 + \frac{x}{1!} (0) + \frac{x^2}{2!} \left(-\frac{1}{2}\right) + \frac{x^3}{3!} (0) + \dots$$

$$\Rightarrow \log(1+\cos x) = \log 2 - \frac{x^2}{4} - \dots$$

7. Expand $y = \log(\sec x + \tan x)$ using Maclaurin's series;

$$\Rightarrow \text{let, } y = \log(\sec x + \tan x)$$

$$\Rightarrow y = \log(\sec x + \tan x) = \log(\sec 0 + \tan 0) = \log 1 = 0$$

$$\Rightarrow y'(x) = \frac{1}{\sec x + \tan x} [\sec x \cdot \tan x + \sec^2 x]$$

$$= \frac{\sec x \cdot \tan x + \sec^2 x}{\sec x + \tan x}$$

$$= \frac{\sec [\cancel{\sec x} + \tan x]}{\cancel{\sec x} + \tan x}$$

$$= \sec x$$

$$= \sec 0 = 1$$

$$\Rightarrow y''(x) = \sec x \cdot \tan x$$

$$= \sec 0 \cdot \tan 0 = 1 + 0 = 0$$

$$\begin{aligned}
 \Rightarrow y'''(x) &= \sec x \cdot \tan x \cdot \tan x + \sec x \cdot \sec^2 x \\
 &= \sec x \cdot \tan^2 x + \sec^3 x \\
 &= \sec x (\sec^2 x - 1) + \sec^3 x \\
 &= \sec^3 x - \sec x + \sec^3 x \\
 &= 2\sec^3 x - \sec x \\
 &= 2\sec^3(0) - \sec(0) = 2 - 1 = 1
 \end{aligned}$$

WKT,

$$\Rightarrow y(x) = y(0) + \frac{x}{1!} y'(0) + \frac{x^2}{2!} y''(0) + \frac{x^3}{3!} y'''(0) + \dots$$

$$\Rightarrow \log(\sec x + \tan x) = 0 + \frac{x}{1!} (1) + \frac{x^2}{2!} (0) + \frac{x^3}{3!} (1) + \frac{x^4}{4!} (0) + \dots$$

$$\Rightarrow \log(\sec x + \tan x) = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

8. Expand $y = \log(\sec x)$ using Maclaurin's expansion upto the term containing x^4 .

$$\Rightarrow \text{let, } y = \log(\sec x)$$

$$f(y) = \log(\sec x) = \log(\sec 0) = \log 1 = 0$$

$$\begin{aligned}
 \Rightarrow y'(x) &= \frac{1}{\sec x} [\sec x \cdot \tan x] \\
 &= \frac{\sec x \cdot \tan x}{\sec x} = \tan x = \tan 0 = 0 \\
 &\quad [y'] \quad [y_0']
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow y''(x) &= \sec^2 x \\
 &= 1 + \tan^2 x \\
 &= 1 + (y')^2 \\
 &= 1 + y_0'^2 = 1 + 0 = 1
 \end{aligned}$$

$$\Rightarrow y'''(x) = 0 + 2y' \cdot y'' = 2y' \cdot y''$$

$$y'''(0) = 2y'_0 \cdot y''_0 = 2(0)(1) = 0$$

$$\Rightarrow y^{iv}(x) = 2[y' y''' + y'' \cdot y'']$$

$$= 2[y' y''' + (y'')^2]$$

$$= 2[y'_0 y'''_0 + (y''_0)^2]$$

$$y^{iv}(0) = 2[0 + (1)^2] = 2$$

WKT,

$$\Rightarrow y(x) = y(0) + \frac{x}{1!} y'(0) + \frac{x^2}{2!} y''(0) + \dots$$

$$\Rightarrow \log(\sec x) = 0 + \frac{x}{1!} (0) + \frac{x^2}{2!} (1) + \frac{x^3}{3!} (0) + \frac{x^4}{4!} (2) + \dots$$

$$\Rightarrow \log(\sec x) = \frac{x^2}{2!} + \frac{2x^4}{4!} + \dots$$

9. $y = \tan x$

$$\Rightarrow \text{Let, } y = \tan x$$

$$y = \tan x \Rightarrow f(0) = \tan(0) = 0$$

$$\Rightarrow y'(x) = \sec^2 x \Rightarrow \sec^2(0) =$$

$$= 1 + \tan^2 x$$

$$= 1 + y^2$$

$$y'(0) = 1 + y_0^2 = 1 + 0 = 1$$

$$\Rightarrow y''(x) = 0 + 2y \cdot y'$$

$$= 2y_0 y'_0 = 2(0)(1) = 0$$

$$\Rightarrow y'''(x) = 2[y \cdot y'' + (y')^2]$$

$$= 2[y_0 y''_0 + (y'_0)^2]$$

$$= 2[0 + 1] = 2$$

WKT,

$$\Rightarrow y(x) = y(0) + \frac{x}{1!} y'(0) + \frac{x^2}{2!} y''(0) + \frac{x^3}{3!} y'''(0) + \dots$$

$$\Rightarrow \tan x = 0 + \frac{x}{1!} (1) + \frac{x^2}{2!} (0) + \frac{x^3}{3!} (2) + \dots$$

$$\Rightarrow \tan x = \frac{x}{1!} + \frac{2x^3}{3!} + \dots$$

INDETERMINATE FORMS:

An indeterminate form involves two functions whose limit cannot be determined and which are of the form.

$$\frac{0}{0}, \frac{\infty}{\infty}, \infty + \infty, \infty - \infty, 0 \times \infty, 0^0, 1^\infty, \infty^0$$

To evaluate these type of expressions we can use "L-Hospital Rule" which is a general method of evaluating indeterminate forms such as $\frac{0}{0}$ (or) $\frac{\infty}{\infty}$.

To evaluate the limits of indeterminate forms for the derivatives in calculus, LH rule can be applied more than once.

Let $f(x)$ & $g(x)$ be the 2 functions, then the indeterminate form of these will be in the form of:-

$$\Rightarrow \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{0}{0} \text{ --- (1)}$$

As per LH rule differentiate both $f(x)$ and $g(x)$ separately and apply the limit value until to get the determinate value.

$$\therefore L = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = \frac{0}{0} \quad (4b)$$

$$\textcircled{1} \Rightarrow L = \lim_{x \rightarrow a} \frac{f''(x)}{g''(x)} = \frac{0}{0} \quad (4) \text{ --- } \textcircled{2}$$

$$\textcircled{2} \Rightarrow L = \lim_{x \rightarrow a} \frac{f'''(x)}{g'''(x)} = K$$

Examples:-

1. Evaluate the following indeterminate forms:-

i) $\lim_{x \rightarrow 0} \frac{\sin x}{x}$

$$\Rightarrow L = \lim_{x \rightarrow 0} \frac{\sin x}{x} = \frac{0}{0} \text{ --- } \textcircled{1}$$

Apply LH rule

$$\textcircled{1} \Rightarrow L = \lim_{x \rightarrow 0} \frac{\cos x}{1}$$

$$L = \frac{\cos 0}{1} = \frac{1}{1} = 1$$

ii) $\lim_{x \rightarrow 0} \frac{\tan x}{x}$

$$\Rightarrow L = \lim_{x \rightarrow 0} \frac{\tan x}{x} = \frac{0}{0} \text{ --- } \textcircled{2}$$

Apply LH rule

$$\textcircled{2} \Rightarrow L = \lim_{x \rightarrow 0} \frac{\sec^2 x}{1}$$

$$L = \frac{\sec^2(0)}{1} = \frac{1^2}{1} = 1$$

iii) $\lim_{x \rightarrow 0} \frac{a^x - b^x}{x}$

$$\Rightarrow L = \lim_{x \rightarrow 0} \frac{a^x - b^x}{x} = \frac{1-1}{0} = \frac{0}{0} \text{ --- } \textcircled{3}$$

apply LH rule

$$\Rightarrow L = \lim_{x \rightarrow 0} \frac{a^x \log a - b^x \log b}{1}$$

$$= \frac{a^0 \log a - b^0 \log b}{1}$$

$$= \log a - \log b$$

$$\Rightarrow L = \log \left(\frac{a}{b} \right)$$

Problems:-

Evaluate the following indeterminate forms:-

$$1. \lim_{x \rightarrow 0} \left[\frac{a^x + b^x}{2} \right]^{1/x}$$

$$\Rightarrow \text{let, } L = \lim_{x \rightarrow 0} \left[\frac{a^x + b^x}{2} \right]^{1/x} = 1^\infty$$

Taking log on b.s

$$\textcircled{1} \Rightarrow \log L = \lim_{x \rightarrow 0} \log \left[\frac{a^x + b^x}{2} \right]^{1/x}$$

$$\Rightarrow \log L = \lim_{x \rightarrow 0} \frac{1}{x} \cdot \log \left[\frac{a^x + b^x}{2} \right]$$

$$\Rightarrow \log L = \lim_{x \rightarrow 0} \frac{1}{x} [\log(a^x + b^x) - \log 2]$$

$$\Rightarrow \log L = \lim_{x \rightarrow 0} \frac{\log(a^x + b^x) - \log 2}{x} = \frac{0}{0} \quad \text{---} \textcircled{2}$$

apply LH rule

$$\textcircled{2} \Rightarrow \log L = \lim_{x \rightarrow 0} \frac{\frac{1}{a^x + b^x} [a^x \log a - b^x \log b]}{1} = 0$$

$$\Rightarrow \log l = \lim_{x \rightarrow 0} \frac{a^x \log a - b^x \log b}{a^x + b^x}$$

$$\Rightarrow \log l = \frac{\log a - \log b}{2}$$

$$\Rightarrow \log l = \frac{1}{2} \log(ab)$$

$$\Rightarrow \log l = \log(ab)^{1/2}$$

$$\Rightarrow l = (ab)^{1/2}$$

$$\Rightarrow l = \sqrt{ab}$$

$$2. \lim_{x \rightarrow 0} \left[\frac{a^x + b^x + c^x}{3} \right]^{1/x}$$

$$\Rightarrow \text{Let, } l = \lim_{x \rightarrow 0} \left[\frac{a^x + b^x + c^x}{3} \right]^{1/x}$$

Taking log on b.s

$$\Rightarrow \log l = \lim_{x \rightarrow 0} \log \left[\frac{a^x + b^x + c^x}{3} \right]^{1/x}$$

$$\Rightarrow \log l = \lim_{x \rightarrow 0} \frac{1}{x} \cdot \log \left[\frac{a^x + b^x + c^x}{3} \right]$$

$$\Rightarrow \log l = \lim_{x \rightarrow 0} \frac{1}{x} \log [a^x + b^x + c^x] - \log 3$$

apply LH rule

$$\Rightarrow \log l = \lim_{x \rightarrow 0} \frac{\frac{1}{a^x + b^x + c^x} [a^x \log a + b^x \log b + c^x \log c]}{x} = \frac{0}{0}$$

apply LH rule.

$$\textcircled{2} \Rightarrow \log l = \lim_{x \rightarrow 0} \frac{a^x \log a + b^x \log b + c^x \log c}{a^x + b^x + c^x}$$

$$\Rightarrow \log l = \frac{\log a + \log b + \log c}{3}$$

$$\Rightarrow \log L = \frac{1}{3} \log(a, b, c)$$

$$\Rightarrow \log L = \log(abc)^{1/3}$$

$$\Rightarrow L = (abc)^{1/3}$$

$$3. \lim_{x \rightarrow 0} \left[\frac{a^x + b^x + c^x + d^x}{4} \right]^{1/x}$$

$$\Rightarrow \text{let, } L = \lim_{x \rightarrow 0} \left[\frac{a^x + b^x + c^x + d^x}{4} \right]^{1/x}$$

taking log on b.s

$$\Rightarrow \log L = \lim_{x \rightarrow 0} \log \left[\frac{a^x + b^x + c^x + d^x}{4} \right]^{1/x}$$

$$\Rightarrow \log L = \lim_{x \rightarrow 0} \frac{1}{x} \log \left[\frac{a^x + b^x + c^x + d^x}{4} \right]$$

$$\Rightarrow \log L = \lim_{x \rightarrow 0} \frac{1}{x} \log [a^x + b^x + c^x + d^x] - \log 4$$

$$\Rightarrow \log L = \lim_{x \rightarrow 0} \frac{\log [a^x + b^x + c^x + d^x] - \log 4}{x}$$

apply LH - rule

$$\Rightarrow \log L = \lim_{x \rightarrow 0} \frac{1}{a^x + b^x + c^x + d^x} [a^x \log a + b^x \log b + c^x \log c + d^x \log d] - 0$$

$$\Rightarrow \log L = \lim_{x \rightarrow 0} \frac{a^x \log a + b^x \log b + c^x \log c + d^x \log d}{a^x + b^x + c^x + d^x}$$

$$\Rightarrow \log L = \frac{\log a + \log b + \log c + \log d}{4}$$

$$\Rightarrow \log L = \frac{1}{4} \log(abcd)$$

$$\Rightarrow \log L = \log(abcd)^{1/4}$$

$$\Rightarrow L = (abcd)^{1/4}$$

$$4. \lim_{x \rightarrow 0} (\cos x)^{1/x^2}$$

$$\Rightarrow \text{Let, } l = \lim_{x \rightarrow 0} (\cos x)^{1/x^2} = 1^\infty \text{ --- (1)}$$

Taking log on b.s

$$\Rightarrow \log l = \lim_{x \rightarrow 0} \log (\cos x)^{1/x^2}$$

$$\Rightarrow \log l = \lim_{x \rightarrow 0} \frac{1}{x^2} \log (\cos x)$$

$$\Rightarrow \log l = \lim_{x \rightarrow 0} \frac{\log (\cos x)}{x^2} = \frac{0}{0} \text{ --- (2)}$$

apply LH - rule

$$\Rightarrow \log l = \lim_{x \rightarrow 0} \frac{\frac{1}{\cos x} (-\sin x)}{2x}$$

$$\Rightarrow \log l = \lim_{x \rightarrow 0} \frac{-\frac{\sin x}{\cos x}}{2x}$$

$$\Rightarrow \log l = \lim_{x \rightarrow 0} \frac{-\tan x}{2x}$$

$$\Rightarrow \log l = -\frac{1}{2} \lim_{x \rightarrow 0} \frac{\tan x}{x} = \frac{0}{0} \text{ --- (3)}$$

apply LH - rule

$$\Rightarrow \log l = -\frac{1}{2} \lim_{x \rightarrow 0} \frac{\sec^2 x}{1}$$

$$\Rightarrow \log l = -\frac{1}{2} \left[\frac{1}{1} \right]$$

$$\Rightarrow \log_e l = -\frac{1}{2}$$

$$\Rightarrow l = e^{-1/2}$$

$$\Rightarrow l = \frac{1}{e^{1/2}}$$

$$\Rightarrow l = \frac{1}{\sqrt{e}}$$

Evaluate :-

$$1. \lim_{x \rightarrow 0} \left[\frac{1}{x} \right]^{2 \sin x}$$

$$\Rightarrow \text{let, } L = \lim_{x \rightarrow 0} \left[\frac{1}{x} \right]^{2 \sin x}$$

apply log on b.s

$$\Rightarrow \log L = \lim_{x \rightarrow 0} 2 \sin x \log \left[\frac{1}{x} \right]$$

$$\Rightarrow \log L = \lim_{x \rightarrow 0} 2 \sin x \log \left[\frac{1}{x} \right]$$

$$\Rightarrow \log L = 2 \lim_{x \rightarrow 0} \frac{\log(1/x)}{1/\sin x}$$

$$\Rightarrow \log L = 2 \lim_{x \rightarrow 0} \frac{-\log x}{\operatorname{cosec} x}$$

$$\Rightarrow \log L = -2 \lim_{x \rightarrow 0} \frac{\log x}{\operatorname{cosec} x} = \frac{\infty}{\infty} \quad \text{--- (2)}$$

apply LH - rule

$$\Rightarrow \log L = -2 \lim_{x \rightarrow 0} \frac{1/x}{-\cot x \cdot \operatorname{cosec} x}$$

$$\Rightarrow \log L = 2 \lim_{x \rightarrow 0} \frac{1}{\frac{x \cos x}{\sin x} \cdot \frac{1}{\sin x}}$$

$$\Rightarrow \log L = 2 \lim_{x \rightarrow 0} \frac{\sin x}{\cos x \left[\frac{x}{\sin x} \right]}$$

$$\Rightarrow \log L = 2 \lim_{x \rightarrow 0} \frac{\sin x}{\cos x (1)}$$

$$\Rightarrow \log L = 2 \lim_{x \rightarrow 0} \tan x$$

$$\Rightarrow \log L = 2(0) = 0$$

$$\Rightarrow \log_e L = 0 \Rightarrow L = e^0 \Rightarrow L = 1$$

$$2. \lim_{x \rightarrow 0} \left[\frac{\tan x}{x} \right]^{1/x}$$

$$\Rightarrow \text{let, } L = \lim_{x \rightarrow 0} \left[\frac{\tan x}{x} \right]^{1/x} = \infty^\infty \text{ --- (1)}$$

log on b.s

$$\Rightarrow \log L = \lim_{x \rightarrow 0} \log \left[\frac{\tan x}{x} \right]^{1/x}$$

$$\Rightarrow \log L = \lim_{x \rightarrow 0} \frac{1}{x} \log \left[\frac{\tan x}{x} \right]$$

$$\Rightarrow \log L = \lim_{x \rightarrow 0} \frac{\log(\tan x) - \log(x)}{x} = \frac{0}{0} \text{ --- (2)}$$

Apply LH rule

$$\Rightarrow \log L = \lim_{x \rightarrow 0} \frac{\frac{1}{\tan x} (\sec^2 x) - \frac{1}{x}}{1}$$

$$\Rightarrow \log L = \lim_{x \rightarrow 0} \frac{\sec^2 x}{\tan x} - \frac{1}{x}$$

$$\Rightarrow \log L = \lim_{x \rightarrow 0} \frac{1/\cos^2 x}{\sin x / \cos x} - \frac{1}{x}$$

$$\Rightarrow \log L = \lim_{x \rightarrow 0} \frac{1}{\sin x \cdot \cos x} - \frac{1}{x}$$

$$\Rightarrow \log L = \lim_{x \rightarrow 0} \left[\frac{2}{2 \sin x \cdot \cos x} - \frac{1}{x} \right]$$

$$\Rightarrow \log L = \lim_{x \rightarrow 0} \left[\frac{2}{\sin 2x} - \frac{1}{x} \right]$$

$$\Rightarrow \log L = \lim_{x \rightarrow 0} \left[\frac{2x - \sin 2x}{\sin 2x} \right]$$

$$\Rightarrow \log L = \lim_{x \rightarrow 0} \left[\frac{2x - \sin 2x}{2x^2 \left(\frac{\sin 2x}{2x} \right)} \right]$$

$$\Rightarrow \log l = \frac{1}{2} \lim_{x \rightarrow 0} \frac{2x - \sin 2x}{x^2} = \frac{0}{0} \quad \text{--- (3)}$$

$$\Rightarrow \log l = \frac{1}{2} \lim_{x \rightarrow 0} \frac{2 - 2 \cos 2x}{2x}$$

$$\Rightarrow \log l = \frac{1}{2} \lim_{x \rightarrow 0} \frac{1 - \cos 2x}{x} = \frac{0}{0}$$

$$\Rightarrow \log l = \frac{1}{2} \lim_{x \rightarrow 0} \frac{0 - (-2) \sin 2x}{1}$$

$$\Rightarrow \log l = \frac{1}{2} \lim_{x \rightarrow 0} 2 \sin 2x$$

$$\Rightarrow \log l = \lim_{x \rightarrow 0} \sin 2x$$

$$\Rightarrow \log_e l = 0$$

$$\Rightarrow l = e^0$$

$$\Rightarrow l = 1$$

$$3. \lim_{x \rightarrow 0} \left[\frac{\sin x}{x} \right]^{1/x^2}$$

$$\Rightarrow \text{Let, } l = \lim_{x \rightarrow 0} \left[\frac{\sin x}{x} \right]^{1/x^2} = \infty$$

log on b.s

$$\Rightarrow \log l = \lim_{x \rightarrow 0} \log \left[\frac{\sin x}{x} \right]^{1/x^2}$$

$$\Rightarrow \log l = \lim_{x \rightarrow 0} \frac{1}{x^2} \log \left[\frac{\sin x}{x} \right]$$

$$\Rightarrow \log l = \lim_{x \rightarrow 0} \frac{\log(\sin x) - \log(x)}{x^2} = \frac{0}{0}$$

apply LH rule

$$\Rightarrow \log l = \lim_{x \rightarrow 0} \frac{\frac{1}{\sin x} (\cos x) - \frac{1}{x}}{2x}$$

$$\Rightarrow \log l = \lim_{x \rightarrow 0} \frac{\frac{\cos x}{\sin x} - \frac{1}{x}}{2x}$$

$$\Rightarrow \log l = \lim_{x \rightarrow 0} \frac{\frac{x \cos x - \sin x}{x \sin x}}{2x}$$

$$\Rightarrow \log l = \frac{1}{2} \lim_{x \rightarrow 0} \frac{x \cos x - \sin x}{x^2 \sin x}$$

$$\Rightarrow \log l = \frac{1}{2} \lim_{x \rightarrow 0} \frac{x \cos x - \sin x}{x^3 \left[\frac{\sin x}{x} \right]}$$

$$\Rightarrow \log l = \frac{1}{2} \lim_{x \rightarrow 0} \frac{x \cos x - \sin x}{x^3} \quad \text{--- (2)} = \frac{0}{0}$$

apply LH- rule

$$\Rightarrow \log l = \frac{1}{2} \lim_{x \rightarrow 0} \frac{\cos x - x \sin x - \cos x}{3x^2}$$

$$\Rightarrow \log l = \frac{1}{2} \lim_{x \rightarrow 0} \frac{-x \sin x}{x^2}$$

$$\Rightarrow \log l = -\frac{1}{6} \lim_{x \rightarrow 0} \frac{\sin x}{x}$$

$$\Rightarrow \log l = -\frac{1}{6} (1)$$

$$\Rightarrow \log_e l = -\frac{1}{6}$$

$$\Rightarrow l = e^{-1/6}$$

Partial Differentiation

Let, $z = f(x, y)$ be a real number function defined on any other interval, then the first and second order partial derivatives can be denoted as:-

→ First order Partial Differentiation:-

$$1. \frac{\partial z}{\partial x} = \frac{\partial t}{\partial x} = z_x = f_x = p$$

$$\frac{\partial z}{\partial y} = \frac{\partial t}{\partial y} = z_y = f_y = q$$

→ Second Order Partial Differentiation:-

$$2. \frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 t}{\partial x^2} = z_{xx} = f_{xx} = r$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 t}{\partial y^2} = z_{yy} = f_{yy} = t$$

$$\frac{\partial}{\partial y} \left[\frac{\partial z}{\partial x} \right] = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 t}{\partial y \partial x} = z_{yx} = f_{yx} = s$$

$$\frac{\partial}{\partial x} \left[\frac{\partial z}{\partial y} \right] = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 t}{\partial x \partial y} = z_{xy} = f_{xy} = s$$

$$\therefore \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}$$

Problems :-

1. Find the first and second order, the partial derivatives of ;

$$\text{if } z = xy^2 + x^2y$$

$$\Rightarrow z = xy^2 + x^2y \text{ ————— ①}$$

diff partially w.r to 'x'

$$\Rightarrow \frac{\partial z}{\partial x} = y^2 \frac{\partial}{\partial x}(x) + y \frac{\partial}{\partial x}(x^2)$$

$$\Rightarrow \frac{\partial z}{\partial x} = y^2(1) + y(2x)$$

$$\Rightarrow \frac{\partial z}{\partial x} = y^2 + 2xy \text{ ————— ②}$$

Similarly,

$$\Rightarrow \frac{\partial z}{\partial y} = x \cdot \frac{\partial}{\partial y}(y^2) + x^2 \frac{\partial}{\partial y}(y)$$

$$\Rightarrow \frac{\partial z}{\partial y} = 2xy + x^2 \text{ ————— ③}$$

diff ② w.r to (x)

$$\therefore \frac{\partial}{\partial x} \cdot \frac{\partial z}{\partial x} = y^2 \frac{\partial}{\partial x}(1) + 2y \frac{\partial}{\partial x}(x)$$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2} = y^2(0) + 2y(1)$$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2} = 2y \text{ ————— ④}$$

diff ③ w.r to (y)

$$\therefore \frac{\partial}{\partial y} \cdot \frac{\partial z}{\partial y} = 2x \frac{\partial}{\partial y}(y) + x^2 \frac{\partial}{\partial y}(1)$$

$$\Rightarrow \frac{\partial^2 z}{\partial y^2} = 2x(1) + 0$$

$$\Rightarrow \frac{\partial^2 z}{\partial y^2} = 2x \quad \text{--- (5)}$$

diff eqn (3) w.r to (x)

$$\therefore \frac{\partial}{\partial x} \cdot \frac{\partial z}{\partial y} = 2y \cdot \frac{\partial}{\partial x} (x) + \frac{\partial}{\partial x} (x^2)$$

$$\Rightarrow \frac{\partial^2 z}{\partial x \partial y} = 2y + 2x$$

diff eqn (2) w.r to (y)

$$\therefore \frac{\partial}{\partial y} \cdot \frac{\partial z}{\partial x} = \frac{\partial}{\partial y} (y^2) + 2x \cdot \frac{\partial}{\partial y} (y)$$

$$\Rightarrow \frac{\partial^2 z}{\partial y \partial x} = 2y + 2x$$

ii) $z = x^2 \sin y - y^2 \sin x$

$$\Rightarrow z = x^2 \sin y - y^2 \sin x \quad \text{--- (1)}$$

diff eqn (1) partially w.r to x

$$\therefore \frac{\partial z}{\partial x} = \sin y \cdot \frac{\partial}{\partial x} (x^2) - y^2 \frac{\partial}{\partial x} (\sin x)$$

$$\Rightarrow \frac{\partial z}{\partial x} = 2x \sin y - y^2 \cos x \quad \text{--- (2)}$$

diff eqn (1) w.r to y

$$\Rightarrow \frac{\partial z}{\partial y} = x^2 \frac{\partial}{\partial y} (\sin y) - \sin x \cdot \frac{\partial}{\partial y} (y^2)$$

$$\Rightarrow \frac{\partial z}{\partial y} = x^2 \cos y - 2y \sin x \quad \text{--- (3)}$$

$$\Rightarrow \frac{\partial}{\partial x} \cdot \frac{\partial z}{\partial x} = \frac{\partial^2 z}{\partial x^2} = 2 \sin y \cdot \frac{\partial}{\partial x} (x) - y^2 \frac{\partial}{\partial x} (\cos x)$$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2} = 2 \sin y + y^2 \sin x \quad \text{--- (4)}$$

diff (3) w.r to y

$$\Rightarrow \frac{\partial}{\partial y} \cdot \frac{\partial z}{\partial y} = \frac{\partial^2 z}{\partial y^2} = x^2 \frac{\partial}{\partial y} (\cos y) - 2 \sin x \cdot \frac{\partial}{\partial y} (y)$$

$$\Rightarrow \frac{\partial^2 z}{\partial y^2} = -x^2 \sin y - 2 \sin x$$

diff (3) w.r to x

$$\therefore \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x} \cdot \frac{\partial z}{\partial y} = \cos y \cdot \frac{\partial}{\partial x} (x^2) - 2y \frac{\partial}{\partial x} (\sin x)$$

$$= 2x \cos y - 2y \cos x$$

2. If $u = x^3 - 3xy^2 + x + e^x \cos y + 1$

Show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$$\Rightarrow u = x^3 - 3xy^2 + x + e^x \cos y + 1 \quad \text{--- (1)}$$

diff w.r to (x) partially

$$\Rightarrow \frac{\partial u}{\partial x} = 3x^2 - 3y^2 (1) + (1) + e^x \cos y + 0$$

$$\Rightarrow \frac{\partial u}{\partial x} = 3x^2 - 3y^2 + e^x \cos y + 1 \quad \text{--- (2)}$$

$$\therefore \frac{\partial}{\partial x} \cdot \frac{\partial u}{\partial x} = 6x - 0 + e^x \cos y + 0$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = 6x + e^x \cos y \quad \text{--- (3)}$$

Similarly,

eqn (1) diff w.r to (y)

$$\Rightarrow \frac{\partial u}{\partial y} = 0 - 3x(2y) + 0 + e^x(-\sin y) + 0$$

$$\Rightarrow \frac{\partial u}{\partial y} = -6xy - e^x \sin y$$

$$\therefore \frac{\partial}{\partial y} \cdot \frac{\partial u}{\partial y} = -6x(1) - e^x \cos y$$

$$\Rightarrow \frac{\partial^2 u}{\partial y^2} = -6x - e^x \cos y \quad \text{--- (4)}$$

$\therefore$ LHS,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$\Rightarrow 6x + e^x \cos y - 6x - e^x \cos y$$

$$\Rightarrow 0$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

3. If $r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$

Show that, $\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2} = \frac{2}{r}$

$$\Rightarrow r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2 \quad \text{--- (1)}$$

diff w.r to (x) partially

$$\Rightarrow 2r \cdot \frac{dr}{dx} = 2(x-a)$$

$$\Rightarrow r \cdot \frac{dr}{dx} = (x-a)$$

$$\Rightarrow \frac{dr}{dx} = \frac{(x-a)}{r}$$

$$\Rightarrow \frac{\partial^2 r}{\partial x^2} = \frac{r(1) - (x-a) \frac{\partial r}{\partial x}}{r^2}$$

$$\Rightarrow \frac{\partial^2 r}{\partial x^2} = \frac{1 - (x-a) \left(\frac{x-a}{r} \right)}{r^2}$$

$$\Rightarrow \frac{\partial^2 r}{\partial x^2} = \frac{1 - \frac{(x-a)^2}{r}}{r^2}$$

$$\Rightarrow \frac{\partial^2 r}{\partial x^2} = \frac{1}{r^2} - \frac{(x-a)^2}{r^3}$$

$$\Rightarrow \frac{\partial^2 r}{\partial x^2} = \frac{1}{r} \cdot \frac{(x-a)^2}{r^3} \quad \text{--- (2)}$$

Similarly,

$$\Rightarrow \frac{\partial^2 r}{\partial y^2} = \frac{1}{r} - \frac{(y-a)^2}{r^3} \quad \text{--- (3)}$$

$$\Rightarrow \frac{\partial^2 r}{\partial z^2} = \frac{1}{r} - \frac{(z-a)^2}{r^3} \quad \text{--- (4)}$$

LHS,

$$\begin{aligned} & \frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2} \\ &= \frac{1}{r} - \frac{(x-a)^2}{r^3} + \frac{1}{r} - \frac{(y-a)^2}{r^3} + \frac{1}{r} - \frac{(z-a)^2}{r^3} \\ &= \frac{3}{r} - \frac{1}{r^3} \left[(x-a)^2 + (y-a)^2 + (z-a)^2 \right] \\ &= \frac{3}{r} - \frac{1}{r^3} [r^2] \\ &= \frac{3}{r} - \frac{1}{r} \\ &= \frac{2}{r}, \quad \text{RHS} \end{aligned}$$

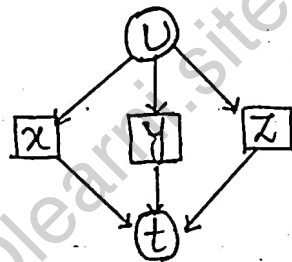
Total Derivatives :-

1. If $u = f(x, y, z)$ be a function in the independent variables of x, y, z then, the total derivatives of u with respect to x, y, z can be defined as,

$$\partial u = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz$$

2. Let, $u = f(x, y, z)$, $x = x(t)$, $y = y(t)$, $z = z(t)$, then the total derivative of ' u ' with respect to ' t ' can be defined as;

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt}$$



3. Let $u = f(p, q, r)$ where, $p = p(x, y, z)$, $q = q(x, y, z)$, $r = r(x, y, z)$ then,

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial x} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial x} + \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial y} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial y} + \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial z} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial z} + \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial z}$$

1. Find the $\frac{du}{dt}$ for the function $u = xy^2 + x^2y$ where, $x = at$, $y = 2at$ and verify the same by the direct method.

$$\Rightarrow u = xy^2 + x^2y \quad \text{--- (1)}$$

where, $x = at$, $y = 2at$

$$\therefore \frac{\partial u}{\partial x} = y^2(1) + (2x)y$$

$$\Rightarrow \frac{\partial u}{\partial x} = y^2 + 2xy$$

$$\therefore \frac{\partial u}{\partial y} = x(2y) + x^2(1)$$

$$\Rightarrow \frac{\partial u}{\partial y} = 2xy + x^2$$

$$\therefore \frac{dx}{dt} = a, \quad \frac{dy}{dt} = 2a$$

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

$$\Rightarrow \frac{du}{dt} = [y^2 + 2xy][2a] + [2xy + x^2][2a]$$

$$\Rightarrow \frac{du}{dt} = a[(2at)^2 + 2(at)(2at)] + 2a[2(at)(2at) + (at)^2]$$

$$\Rightarrow \frac{du}{dt} = a[4a^2t^2 + 4a^2t^2] + 2a[4a^2t^2 + a^2t^2]$$

$$\Rightarrow \frac{du}{dt} = a[8a^2t^2] + 2a[5a^2t^2]$$

$$\Rightarrow \frac{du}{dt} = 8a^3t^2 + 10a^3t^2$$

$$\Rightarrow \frac{du}{dt} = 18a^3t^2$$

Direct method,

$$\therefore (1) \Rightarrow u = (at)(2at)^2 + (at)^2(2at)$$

$$u = (at)(4a^2t^2) + (a^2t^2)(2at)$$

$$u = 4a^3t^3 + 2a^3t^3$$

$$u = 6a^3t^3$$

$$\therefore \frac{du}{dt} = 6a^3(3t^2)$$

$$\Rightarrow \frac{du}{dt} = 18a^3t^2$$

2. Find $\frac{du}{dt}$ for the function $u = xy + yz + zx$ with
 $x = t \cos t$, $y = t \sin t$, $z = t$ at $t = \pi/4$

$$\Rightarrow u = xy + yz + zx \quad \text{--- (1)}$$

where, $x = t \cos t$, $y = t \sin t$, $z = t$

$$\therefore \frac{\partial u}{\partial z} = y + 0 + x = y + x$$

$$\Rightarrow \frac{\partial u}{\partial z} = t \sin t + t$$

$$\therefore \frac{\partial u}{\partial y} = x + z + 0 = x + z$$

$$\Rightarrow \frac{\partial u}{\partial y} = t \cos t + t$$

$$\therefore \frac{\partial u}{\partial x} = 0 + y + z = y + z$$

$$\Rightarrow \frac{\partial u}{\partial x} = t \sin t + t \cos t$$

$$\therefore \frac{dx}{dt} = 1 \cdot \cos t + t(-\sin t) = \cos t - t \sin t$$

$$\frac{dy}{dt} = 1 \cdot \sin t + t \cos t = \sin t + t \cos t$$

$$\frac{dz}{dt} = 1$$

$$\therefore \frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt}$$

$$\Rightarrow \frac{du}{dt} = (t \sin t + t) (\cos t - t \sin t) + (t \cos t + t) (\sin t + t \cos t) + (\sin t + t \cos t)$$

$$\text{at } t = \pi/4$$

$$\Rightarrow \frac{du}{dt} = \left[\frac{\pi}{4\sqrt{2}} + \frac{\pi}{4} \right] \left[\frac{1}{\sqrt{2}} - \frac{\pi}{4\sqrt{2}} \right] + \left[\frac{\pi}{4\sqrt{2}} + \frac{\pi}{4} \right] \left[\frac{1}{\sqrt{2}} + \frac{\pi}{4\sqrt{2}} \right] + \left[\frac{\pi}{4\sqrt{2}} + \frac{\pi}{4\sqrt{2}} \right]$$

$$= \left[\frac{\pi}{4\sqrt{2}} + \frac{\pi}{4} \right] \left[\frac{1}{\sqrt{2}} - \frac{\pi}{4\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{\pi}{4\sqrt{2}} \right] + \frac{2\pi}{4\sqrt{2}}$$

$$= \left[\frac{\pi}{4\sqrt{2}} + \frac{\pi}{4} \right] \left[\frac{2}{\sqrt{2}} \right] + \frac{\pi}{2\sqrt{2}}$$

$$= \left[\frac{\pi}{4\sqrt{2}} + \frac{\pi}{4} \right] \sqrt{2} + \frac{\pi}{2\sqrt{2}}$$

$$= \frac{\pi}{4} + \frac{\pi}{4} \sqrt{2} + \frac{\pi}{2\sqrt{2}}$$

$$= \frac{\pi}{4} + \frac{\sqrt{2}\pi}{2\sqrt{2}\sqrt{2}} + \frac{\pi}{2\sqrt{2}}$$

$$= \frac{\pi}{4} + \frac{2\pi}{2\sqrt{2}}$$

$$= \frac{\pi}{4} + \frac{\pi}{\sqrt{2}}$$

3. Find $\frac{du}{dt}$ for the function $u = e^{x^2+y^2+z^2}$ where,

$$x = t^2 + 1, \quad y = t \cos t, \quad z = \sin t \quad \text{at } t = 0.$$

$$\Rightarrow u = e^{x^2+y^2+z^2} \quad \text{--- (1)}$$

$$\text{where, } x = t^2 + 1, \quad y = t \cos t, \quad z = \sin t$$

$$\Rightarrow \frac{\partial u}{\partial x} = 2x e^{x^2+y^2+z^2}$$

$$\Rightarrow \frac{\partial u}{\partial x} = 2(t^2+1) e^{(t^2+1)^2 + (t^2 \cos^2 t + \sin^2 t)}$$

$$\Rightarrow \frac{\partial u}{\partial y} = 2y \cdot e^{x^2+y^2+z^2}$$

$$\Rightarrow \frac{\partial u}{\partial y} = 2t \cos t \cdot e^{(t^2+1)^2 + (t^2 \cos^2 t + \sin^2 t)}$$

$$\Rightarrow \frac{\partial u}{\partial z} = 2z \cdot e^{x^2+y^2+z^2}$$

$$\Rightarrow \frac{\partial u}{\partial z} = 2 \sin t \cdot e^{(t^2+1)^2 + (t^2 \cos^2 t + \sin^2 t)}$$

$$\therefore \frac{dx}{dt} = 2t$$

$$\frac{dy}{dt} = \cos t - t \sin t$$

$$\frac{dz}{dt} = \cos t$$

$$\therefore \frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt}$$

$$\Rightarrow \frac{du}{dt} = \left[2(t^2+1) e^{(t^2+1)^2 + (t^2 \cos^2 t + \sin^2 t)} \right] (2t) + \left[2t \cos t e^{(t^2+1)^2 + (t^2 \cos^2 t + \sin^2 t)} \right] (\cos t - t \sin t) + \left[2 \sin t e^{(t^2+1)^2 + (t^2 \cos^2 t + \sin^2 t)} \right] (\cos t)$$

$$\Rightarrow \left[\frac{du}{dt} \right]_{t=0} = 0$$

4. If $u = f(y-z, z-x, x-y)$ show that,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

$\Rightarrow$ Let, $p = y-z$, $q = z-x$, $r = x-y$

then, $u = f(p, q, r)$ — (2)

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial x} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial x} + \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} (0) + \frac{\partial u}{\partial q} (-1) + \frac{\partial u}{\partial r} (1)$$

$$\Rightarrow \frac{\partial u}{\partial x} = -\frac{\partial u}{\partial q} + \frac{\partial u}{\partial r} \text{ — (3)}$$

$$\therefore \frac{\partial u}{\partial y} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial y} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial y} + \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y}$$

$$\Rightarrow \frac{\partial u}{\partial y} = \frac{\partial u}{\partial p} (1) + \frac{\partial u}{\partial q} (0) + \frac{\partial u}{\partial r} (-1)$$

$$\Rightarrow \frac{\partial u}{\partial y} = \frac{\partial u}{\partial p} - \frac{\partial u}{\partial r} \text{ — (4)}$$

$$\therefore \frac{\partial u}{\partial z} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial z} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial z} + \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial z}$$

$$\Rightarrow \frac{\partial u}{\partial z} = \frac{\partial u}{\partial p} (-1) + \frac{\partial u}{\partial q} (1) + \frac{\partial u}{\partial r} (0)$$

$$\Rightarrow \frac{\partial u}{\partial z} = -\frac{\partial u}{\partial p} + \frac{\partial u}{\partial q} \text{ — (5)}$$

$$\textcircled{3} + \textcircled{4} + \textcircled{5} \Rightarrow \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$$

$$\Rightarrow -\frac{\partial u}{\partial q} + \frac{\partial u}{\partial r} + \frac{\partial u}{\partial p} - \frac{\partial u}{\partial r} - \frac{\partial u}{\partial p} + \frac{\partial u}{\partial q}$$

$$\Rightarrow 0$$

5. If $u = f(2x-3y, 3y-4z, 4z-2x)$ show that,
 $6u_x + 4u_y + 3u_z = 0$ (or)

$$\frac{1}{2} u_x + \frac{1}{3} u_y + \frac{1}{4} u_z = 0$$

$\Rightarrow$ Let, $p = 2x-3y$, $q = 3y-4z$, $r = 4z-2x$

$$\therefore \textcircled{1} \Rightarrow u = f(p, q, r) \text{ --- (2)}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial x} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial x} + \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} (2) + \frac{\partial u}{\partial q} (0) + \frac{\partial u}{\partial r} (-2)$$

$$\Rightarrow \frac{\partial u}{\partial x} = 2 \frac{\partial u}{\partial p} - 2 \frac{\partial u}{\partial r}$$

$$\Rightarrow \frac{1}{2} \frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} - \frac{\partial u}{\partial r} \text{ --- (3)}$$

$$\therefore \frac{\partial u}{\partial y} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial y} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial y} + \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y}$$

$$\Rightarrow \frac{\partial u}{\partial y} = \frac{\partial u}{\partial p} (-3) + \frac{\partial u}{\partial q} (3) + \frac{\partial u}{\partial r} (0)$$

$$\Rightarrow \frac{1}{3} \frac{\partial u}{\partial y} = -\frac{\partial u}{\partial p} + \frac{\partial u}{\partial q} \text{ --- (4)}$$

$$\therefore \frac{\partial u}{\partial z} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial z} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial z} + \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial z}$$

$$\Rightarrow \frac{\partial u}{\partial z} = \frac{\partial u}{\partial p} (0) + \frac{\partial u}{\partial q} (-4) + \frac{\partial u}{\partial r} (4)$$

$$\Rightarrow \frac{1}{4} \frac{\partial u}{\partial z} = -\frac{\partial u}{\partial q} + \frac{\partial u}{\partial r} \text{ --- (5)}$$

(3) + (4) + (5)

$$\Rightarrow \frac{1}{2} \frac{\partial u}{\partial x} + \frac{1}{3} \frac{\partial u}{\partial y} + \frac{1}{4} \frac{\partial u}{\partial z}$$

$$\Rightarrow \frac{\cancel{\partial u}}{\cancel{\partial p}} - \frac{\cancel{\partial u}}{\cancel{\partial r}} - \frac{\cancel{\partial u}}{\cancel{\partial p}} + \frac{\cancel{\partial u}}{\cancel{\partial r}} - \frac{\cancel{\partial u}}{\cancel{\partial q}} + \frac{\cancel{\partial u}}{\cancel{\partial r}} = 0$$

$$\Rightarrow \frac{1}{2} U_x + \frac{1}{3} U_y + \frac{1}{4} U_z = 0$$

$$\Rightarrow \frac{6U_x + 4U_y + 3U_z}{12} = 0$$

$$\Rightarrow 6U_x + 4U_y + 3U_z = 0$$

6. If $u = f\left(\frac{y-x}{xy}, \frac{z-x}{zx}\right)$ show that,

$$x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} + z^2 \frac{\partial u}{\partial z} = 0$$

$$\Rightarrow \text{Let, } p = \frac{y-x}{xy}, \quad q = \frac{z-x}{zx}$$

$$\text{then, } u = f(p, q) \quad \text{--- (1)}$$

$$\therefore \frac{\partial p}{\partial x} = \frac{xy(-1) - (y-x)(y)}{(xy)^2}$$

$$\Rightarrow \frac{\partial p}{\partial x} = \frac{-xy - y^2 + xy}{(xy)^2}$$

$$\Rightarrow \frac{\partial p}{\partial x} = \frac{-y^2}{x^2 y^2} = -\frac{1}{x^2}$$

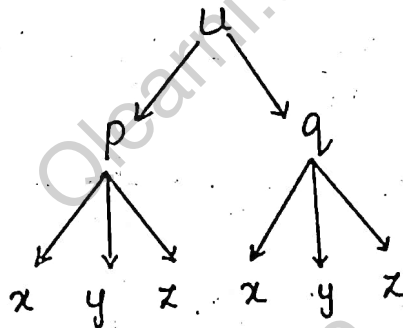
$$\therefore \frac{\partial p}{\partial y} = \frac{xy(1) - (y-x)(x)}{(xy)^2}$$

$$\Rightarrow \frac{\partial p}{\partial y} = \frac{xy - xy + x^2}{x^2 y^2}$$

$$\Rightarrow \frac{\partial p}{\partial y} = \frac{x^2}{x^2 y^2}$$

$$\Rightarrow \frac{\partial p}{\partial y} = \frac{1}{y^2}$$

$$\therefore \frac{\partial p}{\partial z} = 0$$



$$\Rightarrow \frac{\partial q}{\partial x} = \frac{zx(-1) - (z-x)(z)}{(zx)^2}$$

$$\Rightarrow \frac{\partial q}{\partial x} = \frac{-zx - z^2 + zx}{z^2 x^2}$$

$$\Rightarrow \frac{\partial q}{\partial x} = \frac{-z^2}{z^2 x^2}$$

$$\Rightarrow \frac{\partial q}{\partial x} = -\frac{1}{x^2}$$

$$\Rightarrow \frac{\partial q}{\partial x} = '0'$$

$$\therefore \frac{\partial q}{\partial z} = \frac{(zx)(1) - (z-x)(x)}{(zx)^2}$$

$$\Rightarrow \frac{\partial q}{\partial z} = \frac{zx - zx + x^2}{z^2 x^2}$$

$$\Rightarrow \frac{\partial q}{\partial z} = \frac{x^2}{z^2 x^2}$$

$$\Rightarrow \frac{\partial q}{\partial z} = \frac{1}{z^2}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial x} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial x}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} \left[-\frac{1}{x^2} \right] + \frac{\partial u}{\partial q} \left[-\frac{1}{x^2} \right]$$

$$\Rightarrow x^2 \frac{\partial u}{\partial x} = -\frac{\partial u}{\partial p} - \frac{\partial u}{\partial q} \quad \text{--- (2)}$$

$$\therefore \frac{\partial u}{\partial y} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial y} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial y}$$

$$\Rightarrow \frac{\partial u}{\partial y} = \frac{\partial u}{\partial p} \left[\frac{1}{y^2} \right] + \frac{\partial u}{\partial q} \quad (0)$$

$$\Rightarrow y^2 \frac{\partial u}{\partial y} = \frac{\partial u}{\partial p} \quad \text{--- (3)}$$

$$\therefore \frac{\partial u}{\partial z} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial z} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial z}$$

$$\Rightarrow \frac{\partial u}{\partial z} = \frac{\partial u}{\partial p} (0) + \frac{\partial u}{\partial q} \left[\frac{1}{z^2} \right]$$

$$\Rightarrow z^2 \frac{\partial u}{\partial z} = \frac{\partial u}{\partial q} \quad \text{--- (4)}$$

$$(2) + (3) + (4) =$$

$$x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} + z^2 \frac{\partial u}{\partial z} = -\frac{\partial u}{\partial p} - \frac{\partial u}{\partial q} + \frac{\partial u}{\partial p} + \frac{\partial u}{\partial q}$$

$$\Rightarrow x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} + z^2 \frac{\partial u}{\partial z} = 0$$

7. If $u = f(x-y, y-z, z-x)$ then show that,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0 \quad (\text{or}) \quad U_x + U_y + U_z = 0$$

$$\Rightarrow u = f(x-y, y-z, z-x) \quad \text{--- (1)}$$

Let, $p = (x-y)$ $q = (y-z)$ $r = (z-x)$ then

$$u = f(p, q, r)$$

$$\therefore \frac{\partial p}{\partial x} = 1, \quad \frac{\partial p}{\partial y} = -1, \quad \frac{\partial p}{\partial z} = 0$$

$$\frac{\partial q}{\partial x} = 0, \quad \frac{\partial q}{\partial y} = 1, \quad \frac{\partial q}{\partial z} = -1$$

$$\frac{\partial r}{\partial x} = -1, \quad \frac{\partial r}{\partial y} = 0, \quad \frac{\partial r}{\partial z} = 1$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} \frac{\partial p}{\partial x} + \frac{\partial u}{\partial q} \frac{\partial q}{\partial x} + \frac{\partial u}{\partial r} \frac{\partial r}{\partial x}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} (1) + \frac{\partial u}{\partial q} (0) + \frac{\partial u}{\partial r} (-1)$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} - \frac{\partial u}{\partial r} \quad \text{--- (2)}$$

$$\therefore \frac{\partial u}{\partial y} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial y} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial y} + \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y}$$

$$\Rightarrow \frac{\partial u}{\partial y} = \frac{\partial u}{\partial p} (-1) + \frac{\partial u}{\partial q} (1) + \frac{\partial u}{\partial r} (0)$$

$$\Rightarrow \frac{\partial u}{\partial y} = -\frac{\partial u}{\partial p} + \frac{\partial u}{\partial q} \quad \text{--- (3)}$$

$$\therefore \frac{\partial u}{\partial z} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial z} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial z} + \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial z}$$

$$\Rightarrow \frac{\partial u}{\partial z} = \frac{\partial u}{\partial p} (0) + \frac{\partial u}{\partial q} (-1) + \frac{\partial u}{\partial r} (1)$$

$$\Rightarrow \frac{\partial u}{\partial z} = -\frac{\partial u}{\partial q} + \frac{\partial u}{\partial r} \quad \text{--- (4)}$$

$$\begin{aligned} (2) + (3) + (4) &\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} \\ &= \frac{\partial u}{\partial p} - \frac{\partial u}{\partial r} - \frac{\partial u}{\partial p} + \frac{\partial u}{\partial q} - \frac{\partial u}{\partial q} + \frac{\partial u}{\partial r} \\ &= 0 \end{aligned}$$

8. If $u = f\left[\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right]$ then, show that

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = 0$$

$$\Rightarrow u = f\left[\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right]$$

$$\text{Let, } p = \frac{x}{y}, \quad q = \frac{y}{z}, \quad r = \frac{z}{x}$$

$$\text{then, } u = f(p, q, r) \quad \text{--- (1)}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial x} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial x} + \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} \cdot \frac{1}{y} + \frac{\partial u}{\partial z} (x) \quad \text{--- (2)}$$

$$\therefore \frac{\partial u}{\partial y} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial y} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial y} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial y}$$

$$\Rightarrow \frac{\partial u}{\partial y} = -\frac{1}{y^2} \frac{\partial u}{\partial p} + \frac{1}{z} \frac{\partial u}{\partial q} \quad \text{--- (3)}$$

$$\therefore \frac{\partial u}{\partial z} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial z} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial z} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial z}$$

$$\Rightarrow \frac{\partial u}{\partial z} = -\frac{1}{z^2} \frac{\partial u}{\partial p} + \frac{1}{x} \frac{\partial u}{\partial q} \quad \text{--- (4)}$$

(2) + (3) + (4)

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{1}{y} \frac{\partial u}{\partial p} + z \frac{\partial u}{\partial z} - \frac{1}{y^2} \frac{\partial u}{\partial p} + \frac{1}{z} \frac{\partial u}{\partial q} - \frac{1}{z^2} \frac{\partial u}{\partial q} + \frac{\partial u}{\partial z} \frac{1}{x}$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = 0$$

Jacobians:

1. Suppose $u = u(x, y)$, $v = v(x, y)$ be the 2 functions in the variables of x & y , then the jacobians of u, v with respect to x, y can be defined as;

$$J = J \left[\begin{matrix} u, v \\ x, y \end{matrix} \right] = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

and its inverse can be defined as;

$$J' = J' \left[\frac{x, y}{u, v} \right] = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

2. Suppose, $u = u(x, y, z)$, $v = v(x, y, z)$, $w = w(x, y, z)$ then, the jacobian of u, v, w with respect to x, y, z can be defined as;

$$J = \frac{\partial(u, v, w)}{\partial(x, y, z)} = J \left[\frac{u, v, w}{x, y, z} \right] = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

and its inverse can be defined as;

$$J' = \frac{\partial(x, y, z)}{\partial(u, v, w)} = J' \left[\frac{u, v, w}{x, y, z} \right] = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

where, $J \& J' = 1$

Questions :-

1. Find the jacobian of u, v with x, y where,

$$u = e^x \cos y, \quad v = e^x \sin y$$

$$\Rightarrow \therefore \frac{\partial u}{\partial x} = e^x \cos y, \quad \frac{\partial u}{\partial y} = -e^x \sin y$$

$$\frac{\partial v}{\partial x} = e^x \sin y, \quad \frac{\partial v}{\partial y} = e^x \cos y$$

$$\therefore J = J \left(\frac{u, v}{x, y} \right) = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{vmatrix}$$

$$= e^x \cdot e^x \begin{vmatrix} \cos y & -\sin y \\ \sin y & \cos y \end{vmatrix}$$

$$= e^{2x} [\cos^2 y + \sin^2 y]$$

$$\frac{\partial(u, v)}{\partial(x, y)} = e^{2x}$$

2. If $u = x^2 + y^2 + z^2$, $v = xy + yz + zx$, $w = x + y + z$
then, find $\frac{\partial(u, v, w)}{\partial(x, y, z)}$

$\Rightarrow$ Given ;

$$u = x^2 + y^2 + z^2$$

$$v = xy + yz + zx$$

$$w = x + y + z$$

$$\therefore U_x = 2x, \quad U_y = 2y, \quad U_z = 2z$$

$$V_x = y+z, \quad V_y = x+z, \quad V_z = y+x$$

$$W_x = 1, \quad W_y = 1, \quad W_z = 1$$

$\therefore$ WKT,

$$\Rightarrow \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} U_x & U_y & U_z \\ V_x & V_y & V_z \\ W_x & W_y & W_z \end{vmatrix}$$

$$= \begin{vmatrix} 2x & 2y & 2z \\ y+z & x+z & y+x \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2 \begin{vmatrix} x & y & z \\ y+z & z+x & x+y \\ 1 & 1 & 1 \end{vmatrix}$$

$$R_2 = R_2 + R_1$$

$$= 2 \begin{vmatrix} x & y & z \\ x+y+z & x+y+z & x+y+z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2(x, y, z) \begin{vmatrix} x & y & z \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2(x, y, z) (0)$$

$$\Rightarrow \frac{\partial(u, v, w)}{\partial(x, y, z)} = 0$$

3. If $u = 3x + 2y - z$, $v = x - 2y + z$, $w = x^2 + 2xy - z$
 then, show that $\frac{\partial(u, v, w)}{\partial(x, y, z)} = 0$

$\Rightarrow$ Given,

$$u = 3x + 2y - z$$

$$v = x - 2y + z$$

$$w = x^2 + 2xy - z$$

$$\therefore u_x = 3 \quad u_y = 2 \quad u_z = -1$$

$$v_x = 1 \quad v_y = -2 \quad v_z = 1$$

$$w_x = 2x + 2y - z \quad w_y = 2x \quad w_z = -1$$

$\therefore$ WKT,

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 2 & -1 \\ 1 & -2 & 1 \\ 2x+2y-z & 2x & -1 \end{vmatrix}$$

$$= 3[2x - 2x] - 2[-x - (2x + 2y - z)] - [2x + 2(2x + 2y - z)]$$

$$= 3(0) - 2[-x - 2x - 2y + z] - [2x + 4x + 4y - 2z]$$

$$= 6x + 4y - 2x - 6x - 4y + 2z$$

$$\Rightarrow \frac{\partial(u, v, w)}{\partial(x, y, z)} = 0$$

4. If $u = x + 3y^2 - z^3$, $v = 4x^2yz$, $w = 2z^2 - xy$
 evaluate $\frac{\partial(u, v, w)}{\partial(x, y, z)}$ at the point $(1, -1, 0)$

$\Rightarrow$ Given,

$$u = x + 3y^2 - z^3$$

$$v = 4x^2yz$$

$$w = 2z^2 - xy$$

$$p = (1, -1, 0)$$

$$\therefore u_x = 1 \Rightarrow (u_x)_p = 1$$

$$u_y = 6y \Rightarrow (u_y)_p = -6$$

$$u_z = -3z^2 \Rightarrow (u_z)_p = 0$$

$$\therefore v_x = 8xyz \Rightarrow (v_x)_p = 0$$

$$v_y = 4x^2z \Rightarrow (v_y)_p = 0$$

$$v_z = 4x^2y \Rightarrow (v_z)_p = -4$$

$$\therefore w_x = -y \Rightarrow (w_x)_p = -(-1) = 1$$

$$w_y = -x \Rightarrow (w_y)_p = -1$$

$$w_z = 4z \Rightarrow (w_z)_p = 0$$

WKT,

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -6 & 0 \\ 0 & 0 & -4 \\ 1 & -1 & 0 \end{vmatrix}$$

$$= 1[0-4] + 6[0+4]$$

$$= -4 + 24 = 20$$

5. If $u = x + y + z$, $v = y + z$, $w = z$, then find $\frac{\partial(u, v, w)}{\partial(x, y, z)}$

$\Rightarrow$ Given;

$$u = x + y + z$$

$$v = y + z$$

$$w = z$$

$$\therefore u_x = 1 \quad u_y = 1 \quad u_z = 1$$

$$v_x = 0 \quad v_y = 1 \quad v_z = 1$$

$$w_x = 0 \quad w_y = 0 \quad w_z = 1$$

$$\Rightarrow \therefore \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 1[1-0] - 1[0-0] + 1[0-0]$$

$$= 1$$

6. If $u = \frac{yz}{x}$, $v = \frac{zx}{y}$, $w = \frac{xy}{z}$, then, show

that $\frac{\partial(u, v, w)}{\partial(x, y, z)} = 4$

$\Rightarrow$ Given,

$$u = \frac{yz}{x}$$

$$v = \frac{zx}{y}$$

$$w = \frac{xy}{z}$$

$$\therefore U_x = -\frac{yz}{x^2}$$

$$U_y = \frac{z}{x}$$

$$U_z = \frac{y}{x}$$

$$V_x = \frac{z}{y}$$

$$V_y = -\frac{zx}{y^2}$$

$$V_z = \frac{x}{y}$$

$$W_x = \frac{y}{z}$$

$$W_y = \frac{x}{z}$$

$$W_z = -\frac{xy}{z^2}$$

WKT,

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} =$$

$$\begin{vmatrix} U_x & U_y & U_z \\ V_x & V_y & V_z \\ W_x & W_y & W_z \end{vmatrix}$$

$$= \begin{vmatrix} -\frac{yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & -\frac{zx}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & -\frac{xy}{z^2} \end{vmatrix}$$

$$= \begin{vmatrix} -\frac{yz}{x^2} & \frac{zx}{x^2} & \frac{yx}{x^2} \\ \frac{zy}{y^2} & -\frac{zx}{y^2} & \frac{yx}{y^2} \\ \frac{yz}{z^2} & \frac{xz}{z^2} & -\frac{xy}{z^2} \end{vmatrix}$$

$$= \frac{1}{x^2 y^2 z^2} \begin{vmatrix} -yz & zx & xy \\ yz & -zx & xy \\ yz & zx & -xy \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= (-1)(1-1) - 1(-1-1) + 1(1+1)$$

$$= 0 - (-2) + (2)$$

$$= 2+2 = 4$$

7. If $x = r \sin \theta (\cos \phi)$; $y = r \sin \theta (\sin \phi)$, $z = r \cos \theta$,
then, show that $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2 \sin \theta$

$\Rightarrow$ Soln: $x = r \sin \theta (\cos \phi)$

$y = r \sin \theta (\sin \phi)$

$z = r \cos \theta$

$$\frac{\partial x}{\partial r} = \sin \theta \cos \phi, \quad \frac{\partial y}{\partial r} = \sin \theta \sin \phi, \quad \frac{\partial z}{\partial r} = \cos \theta$$

$$\frac{\partial x}{\partial \theta} = r \cos \theta \cos \phi, \quad \frac{\partial y}{\partial \theta} = r \cos \theta \sin \phi, \quad \frac{\partial z}{\partial \theta} = -r \sin \theta$$

$$\frac{\partial x}{\partial \phi} = -r \sin \theta \sin \phi, \quad \frac{\partial y}{\partial \phi} = r \sin \theta \cos \phi, \quad \frac{\partial z}{\partial \phi} = 0$$

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin \theta \cos \phi & r \cos \theta \cos \phi & -r \sin \theta \sin \phi \\ \sin \theta \sin \phi & r \cos \theta \sin \phi & r \sin \theta \cos \phi \\ \cos \theta & -r \sin \theta & 0 \end{vmatrix}$$

$$= r^2 \sin \theta \begin{vmatrix} \sin \theta \cos \phi & \cos \theta \cos \phi & -\sin \phi \\ \sin \theta \sin \phi & \cos \theta \sin \phi & \cos \phi \\ \cos \theta & -\sin \theta & 0 \end{vmatrix}$$

$$= r^2 \sin \theta [\sin \theta \cos \phi (0 + \sin \theta \cos \phi) - \cos \theta \cos \phi (0 - \cos \theta \cos \phi) - \sin \phi (-\sin^2 \theta \sin \phi - \cos^2 \theta \sin \phi)]$$

$$\begin{aligned}
 &= r^2 \sin \theta \cdot [\cos^2 \phi (\sin^2 \theta + \cos^2 \theta) + \sin^2 \phi (1)] \\
 &= r^2 \sin \theta [\cos^2 \phi + \sin^2 \phi] \\
 J &= r^2 \sin \theta
 \end{aligned}$$

Maxima and Minima for two variables:

1. Let the given function as $u = f(x, y)$
2. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ and make them equal to zero for maxima or minima that is, $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$.
By solving the above let's have (x_0, y_0) , (x_1, y_1) , (x_2, y_2) are called extreme or end points.
3. Find the second order partial derivatives of $u = f(x, y)$. Say,

$$\frac{\partial^2 f}{\partial x^2} = A, \quad \frac{\partial^2 f}{\partial x \partial y} = B, \quad \frac{\partial^2 f}{\partial y^2} = C$$

and find the status of the end points as given below.

- i. If $AC - B^2 > 0$ and $A < 0$, we say that the function may have the maximum at $p(x_0, y_0)$ and the maximum value is $f(x_0, y_0)$
- ii. If $AC - B^2 > 0$ and $A > 0$, we say that the function may have minimum value at $p(x_0, y_0)$ and the minimum value is $f(x_0, y_0)$

iii) If $AC - B^2 = 0$ (or) $AC - B^2 < 0$ we say that there is no maximum nor minimum at $P(x_0, y_0)$ and those points are said to be saddle points and follow the same verification at all the extreme points.

Problems:-

1. Find the extreme value of the function
 $f(x, y) = x^3 + y^3 - 3x - 12y + 20$

$\Rightarrow$ Given,

$$f(x, y) = x^3 + y^3 - 3x - 12y + 20 \quad \text{--- (1)}$$

diff (1) w.r to (x) partially

$$(1) \Rightarrow \frac{\partial f}{\partial x} = 3x^2 + 0 - 3 + 0 - 0$$

$$\Rightarrow \frac{\partial f}{\partial x} = 3x^2 - 3 \quad \text{--- (2)}$$

Similarly,

$$\Rightarrow \frac{\partial f}{\partial y} = 0 + 3y^2 - 0 - 12 + 0$$

$$\Rightarrow \frac{\partial f}{\partial y} = 3y^2 - 12 \quad \text{--- (3)}$$

From max/min

$$\frac{\partial f}{\partial x} = 0$$

$$\frac{\partial f}{\partial y} = 0$$

$$\Rightarrow 3x^2 - 3 = 0$$

$$3y^2 - 12 = 0$$

$$\Rightarrow x^2 - 1 = 0$$

$$y^2 - 4 = 0$$

$$\Rightarrow x^2 = 1$$

$$y^2 = 4$$

$$\Rightarrow x = \pm 1$$

$$y = \pm 2$$

∴ The extreme points are:

$$(-1, -2) \quad (-1, 2) \quad (1, -2) \quad (1, 2)$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = 6x = A$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = 0 = B$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = 6y = C$$

Point	A	B	C	$AC - B^2$	status
$(-1, -2)$	$-6 < 0$	0	-12	$72 > 0$	max
$(-1, 2)$	$-6 < 0$	0	12	$-72 < 0$	saddle
$(1, -2)$	$6 > 0$	0	-12	$-72 < 0$	saddle
$(1, 2)$	$6 > 0$	0	12	$72 > 0$	min

$$\max f = (-1)^3 + (-2)^3 - 3(-1) - 12(-2) + 20$$

$$\max f = -1 - 8 + 3 + 24 + 20 = 38$$

$$\min f = (1)^3 + (2)^3 - 3(1) - 12(2) + 20$$

$$\min f = 1 + 8 - 3 - 24 + 20 = 2$$

2. Find the extreme points of the function

$$f(x, y) = 2(x^2 - y^2) - x^4 + y^4$$

$$\Rightarrow \text{soln: } f(x, y) = 2(x^2 - y^2) - x^4 + y^4 \quad \text{--- (1)}$$

diff (1) w.r to (x) & (y) partially

$$(1) \Rightarrow \frac{\partial f}{\partial x} = 4x - 4x^3 \quad \text{--- (2)}$$

$$(2) \Rightarrow \frac{\partial f}{\partial y} = -4y + 4y^3 \quad \text{--- (3)}$$

$\therefore$ From max/min

$$\Rightarrow 4x - 4x^3 = 0$$

$$\Rightarrow x - x^3 = 0$$

$$\Rightarrow x(1-x^2) = 0$$

$$\Rightarrow x=0, 1-x^2=0$$

$$\Rightarrow x=0, 1-x^2=0$$

$$\Rightarrow x=0, x=1$$

$$-4y + 4y^3 = 0$$

$$-y + y^3 = 0$$

$$y(-1+y^2) = 0$$

$$y=0, -1+y^2=0$$

$$y=0, -1+y^2=0$$

$$y=0, y=1$$

$\therefore$ The extreme points are:

$(0,0) (0,1) (1,0) (1,1)$

$$\frac{\partial^2 f}{\partial x^2} = 4 - 12x^2 = A$$

$$\frac{\partial^2 f}{\partial x \partial y} = 0 = B$$

$$\frac{\partial^2 f}{\partial y^2} = -4 + 12y^2 = C$$

Points	A	B	C	$AC - B^2$	Status
$(0,0)$	$4 > 0$	0	-4	$-16 < 0$	saddle
$(0,1)$	$4 > 0$	0	8	$32 > 0$	min
$(1,0)$	$-8 < 0$	0	-4	$32 > 0$	max
$(1,1)$	$-8 < 0$	0	8	-64	saddle

$$\max f = 2(1-0) - 1 + 0 = 2 - 1 = 1$$

$$\min f = 2(0-1) - 0 + 1 = -2 + 1 = -1$$

3. Examine the function $f(x, y) = 2 + 2x + 2y - x^2 - y^2$

$\Rightarrow$ Soln:-

$$f(x, y) = 2 + 2x + 2y - x^2 - y^2 \quad \text{---(1)}$$

$$\frac{\partial f}{\partial x} = 2 - 2x \quad \text{---(2)}$$

$$\frac{\partial f}{\partial y} = 2 - 2y \quad \text{---(3)}$$

$\therefore$ for max/min

$$\frac{\partial f}{\partial x} = 0 \quad \frac{\partial f}{\partial y} = 0$$

$$\Rightarrow 2 - 2x = 0$$

$$2 - 2y = 0$$

$$\Rightarrow 1 - x = 0$$

$$1 - y = 0$$

$$\Rightarrow x = 1$$

$$y = 1$$

$$\therefore P = (1, 1)$$

$$\therefore \frac{\partial f}{\partial x^2} = -2 = A$$

$$\frac{\partial^2 f}{\partial x \partial y} = 0 = B$$

$$\frac{\partial^2 f}{\partial y^2} = -2 = C$$

$$\therefore A + P(1, 1)$$

$$AC - B^2 = 4 > 0 \quad \& \quad A < 0$$

$$f(x, y), f_{\max}$$

$$\max f = 2 + 2(1) + 2(1) - 1^2 - 1^2$$

$$\max f = 4$$

4. Examine the function $f(x, y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$

$$\Rightarrow f(x, y) = x^3 + 3xy^2 - 15x^2 - 15y^2 \quad \text{---(1)}$$

$$\frac{\partial f}{\partial x} = 3x^2 + 3y^2 - 30x + 72 \quad \text{---(2)}$$

$$\frac{\partial f}{\partial y} = 6xy - 30y \quad \text{---(3)}$$

For max/min

$$\frac{\partial f}{\partial x} = 0 \quad \& \quad \frac{\partial f}{\partial y} = 0$$

$$\text{and (3)} \Rightarrow 6xy - 30y = 0$$

$$\Rightarrow xy - 5y = 0$$

$$\Rightarrow y(x - 5) = 0$$

$$\Rightarrow y = 0, \quad x - 5 = 0$$

$$\Rightarrow y = 0, \quad x = 5$$

when $y = 0$

$$(4) \Rightarrow x^2 - 10x + 24 = 0$$

$$\Rightarrow x^2 - 4x - 6x + 24 = 0$$

$$\Rightarrow x(x - 4) - 6(x - 4) = 0$$

$$\Rightarrow (x - 4)(x - 6) = 0$$

$$\Rightarrow x = 4, 6$$

$\therefore$ Now points are $(4, 0)$ $(6, 0)$

when $x = 5$

$$(4) \Rightarrow 25 + y^2 - 50 + 24 = 0$$

$$\Rightarrow y^2 - 1 = 0$$

$$\Rightarrow y^2 = 1$$

$$\Rightarrow y = 1$$

the point are $(5, 1)$

$$\frac{\partial^2 f}{\partial x^2} = 6x - 30 = A$$

$$\frac{\partial^2 f}{\partial x \partial y} = 6y = B$$

$$\frac{\partial^2 f}{\partial x \partial y} = 6x - 30 = C$$

Points	A	B	C	$AC - B^2$	Status
(4,0)	$-6 < 0$	0	-6	$36 > 0$	max
(6,0)	$+6 > 0$	0	6	$36 > 0$	min
(5,1)	0	0	0	$-36 < 0$	Saddle

5. Examine the function $f(x, y) = x^4 + y^4 - 2(x - y)^2$

⇒ Soln: $f(x, y) = x^4 + y^4 - 2(x - y)^2$ — (1)

diff (1) partially w.r to x & y

$$(1) \Rightarrow \frac{\partial f}{\partial x} = 4x^3 - 4(x - y) \quad (1)$$

$$\Rightarrow \frac{\partial f}{\partial x} = 4x^3 - 4(x - y) \quad (2)$$

and

$$(1) \Rightarrow \frac{\partial f}{\partial y} = 0 + 4y^3 - 4(x - y)(-1)$$

$$\Rightarrow \frac{\partial f}{\partial y} = 4y^3 + 4(x - y) \quad (3)$$

for max/min

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0$$

$$(3) \Rightarrow 4x^3 - 4(x-y) = 0$$

$$\Rightarrow x^3 - (x-y) = 0$$

$$\Rightarrow x^3 = x-y \quad \text{---(4)}$$

$$(3) \Rightarrow 4y^3 + 4(x-y) = 0$$

$$\Rightarrow y^3 + (x-y) = 0$$

$$\Rightarrow y^3 = -(x-y) \quad \text{---(5)}$$

$$\Rightarrow xy^3 = -x^3$$

$$\Rightarrow x^3 + y^3 = 0$$

$$\Rightarrow (x+y)(x^2 - xy + y^2) = 0$$

$$\Rightarrow x+y=0, \quad x^2 - xy + y^2 = 0$$

$$\Rightarrow x = -y \quad \text{---(6)}$$

$$(8) \Rightarrow y^3 = -|y-y|$$

$$\Rightarrow y^3 = 2y$$

$$\Rightarrow y^3 - 2y = 0$$

$$\Rightarrow y=0, \quad y^2 - 2 = 0$$

$$\Rightarrow y=0, \quad y^2 = 2$$

$$\Rightarrow y=0, \quad y = \sqrt{2}$$

$$\Rightarrow x=0, \quad x = \sqrt{2}$$

The extreme points are $(0,0)$ $(-\sqrt{2}, \sqrt{2})$

$$\therefore \frac{\partial^2 f}{\partial x^2} = 12x^2 - 4 = A$$

$$\frac{\partial^2 f}{\partial x \partial y} = 4 = B$$

$$\frac{\partial^2 f}{\partial y^2} = 12y^2 - 4 = C$$

Points	A	B	C	$AC - B^2$	Status
$(0,0)$	$-4 < 0$	4	-4	0	Saddle
$(-\sqrt{5}, \sqrt{5})$	$20 > 0$	4	20	$384 > 0$	min

6. Examine the function $f(x,y) = 2x^2 + 6xy^2 - 3y^2 - 150x$

$\Rightarrow$ Given,

$$f(x,y) = 2x^2 + 6xy^2 - 3y^2 - 150x \quad \text{---(1)}$$

$$(1) \Rightarrow \frac{\partial f}{\partial x} = 6x^2 + 6y^2 - 150 \quad \text{---(2)}$$

$$(1) \Rightarrow \frac{\partial f}{\partial y} = 12xy - 6y \quad \text{---(3)}$$

for max/min

$$(2) \Rightarrow \frac{\partial f}{\partial x} = 0$$

$$\Rightarrow 6x^2 + 6y^2 - 150 = 0$$

$$\Rightarrow x^2 + y^2 - 25 = 0 \quad \text{---(4)}$$

$$(3) \Rightarrow \frac{\partial f}{\partial y} = 0$$

$$\Rightarrow 12xy - 6y = 0$$

$$\Rightarrow 3y(4x - 2) = 0$$

$$\Rightarrow y = 0, \quad 4x - 2 = 0$$

$$\Rightarrow y = 0, \quad 4x = 2$$

$$\Rightarrow y = 0, \quad x = \frac{2}{4}$$

when $y = 0$

$$(4) \Rightarrow x^2 - 25 = 0$$

$$\Rightarrow x^2 = 25$$

$$\Rightarrow x = 5$$

$$\therefore P = (5, 0)$$

$$\text{when, } x = \frac{34}{4}$$

$$(4) \Rightarrow \left(\frac{34}{4}\right)^2 + y^2 - 25 = 0$$

$$\Rightarrow \frac{9y^2}{16} + y^2 - 25 = 0$$

$$\Rightarrow 25y^2 - 400 = 0$$

$$\Rightarrow y^2 - 16 = 0$$

$$\Rightarrow y^2 = 16$$

$$\Rightarrow y = 4$$

$$\therefore Q = (3, 4)$$

$$\frac{\partial^2 f}{\partial x^2} = 18x = A$$

$$\frac{\partial^2 f}{\partial x \partial y} = 18y = B$$

$$\frac{\partial^2 f}{\partial y^2} = 18x - 18y = C$$

Points	A	B	C	$AC - B^2$	status
(5, 0)	$60 > 0$	0	60	$3600 > 0$	min
(3, 4)	$36 > 0$	48	-36	$-3600 > 0$	Saddle

Additional Problems:-

1. If $u = x + y + z$, $uv = y + z$, $uvw = z$, evaluate $\frac{\partial(x, y, z)}{\partial(u, v, w)}$

$\Rightarrow$ Given,

$$u = x + y + z \quad \text{--- (1)}$$

$$uv = y + z \quad \text{--- (2)}$$

$$uvw = z \quad \text{--- (3)}$$

$$(3) \Rightarrow z = uvw$$

$$(2) \Rightarrow uv = y + uvw$$

$$\Rightarrow y = uv - uvw$$

$$(1) \Rightarrow u = x + uv$$

$$\Rightarrow x = u - uv$$

$$\therefore \frac{\partial x}{\partial u} = 1 - v$$

$$\frac{\partial x}{\partial v} = -u$$

$$\frac{\partial x}{\partial w} = 0$$

$$\frac{\partial y}{\partial u} = v - vw$$

$$\frac{\partial y}{\partial v} = u - uw$$

$$\frac{\partial y}{\partial w} = -uv$$

$$\frac{\partial z}{\partial u} = vw$$

$$\frac{\partial z}{\partial v} = uw$$

$$\frac{\partial z}{\partial w} = uv$$

$\therefore$ WKT,

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} =$$

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} 1-v & -u & 0 \\ v(1-w) & u(1-w) & -uv \\ vw & uw & uv \end{vmatrix}$$

$$= U \cdot UV \begin{vmatrix} 1-V & -1 & 0 \\ V-VW & 1-W & -1 \\ VW & W & 1 \end{vmatrix}$$

$$R_2 = R_2 + R_3$$

$$= U^2 \cdot V \begin{vmatrix} 1-V & -1 & 0 \\ V & 1 & 0 \\ VW & W & 1 \end{vmatrix}$$

$$R_1 = R_1 + R_2$$

$$= U^2 V \begin{vmatrix} 1 & 0 & 0 \\ V & 1 & 0 \\ VW & W & 1 \end{vmatrix}$$

$$= U^2 V (1 - 0)$$

$$= U^2 V$$

2. If $z = \frac{x^2 + y^2}{x + y}$, show that $\left[\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} \right]^2 = 4 \left[1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} \right]$

$\Rightarrow$ Given:-

$$z = \frac{x^2 + y^2}{x + y} \quad \text{--- (1)}$$

$$\therefore \frac{\partial z}{\partial x} = \frac{(x+y)(2x) - (x^2 + y^2)(1)}{(x+y)^2}$$

$$= \frac{2x^2 + 2xy - x^2 - y^2}{(x+y)^2}$$

$$= \frac{x^2 + 2xy - y^2}{(x+y)^2} \quad \text{--- (2)}$$

Similarly,

$$\Rightarrow \frac{\partial z}{\partial y} = \frac{(x+y)(2y) - (x^2+y^2)(1)}{(x+y)^2}$$

$$= \frac{2xy + 2y^2 - x^2 - y^2}{(x+y)^2}$$

$$= \frac{-x^2 + 2xy + y^2}{(x+y)^2}$$

$$\therefore \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = \left[\frac{x^2 + 2xy - y^2}{(x+y)^2} \right] - \left[\frac{-x^2 + 2xy + y^2}{(x+y)^2} \right]$$

$$= \frac{1}{(x+y)^2} [x^2 + 2xy - y^2 + x^2 - 2xy - y^2]$$

$$= \frac{2x^2 - 2y^2}{(x+y)^2}$$

$$= \frac{2(x^2 - y^2)}{(x+y)^2}$$

S.O.B.S

$$\therefore \left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} \right)^2 = \frac{4(x^2 - y^2)^2}{(x+y)^4} \quad \text{--- (3)}$$

Similarly,

$$4 \left[1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} \right] = 4 \left\{ 1 - \left[\frac{x^2 + 2xy - y^2}{(x+y)^2} \right] - \left[\frac{-x^2 + 2xy + y^2}{(x+y)^2} \right] \right\}$$

$$= 4 \left[\frac{(x+y)^2 - (x^2 + 2xy - y^2) - (-x^2 + 2xy + y^2)}{(x+y)^2} \right]$$

$$= 4 \left[\frac{x^2 + 2xy + y^2 - x^2 - 2xy + y^2 + x^2 - 2xy - y^2}{(x+y)^2} \right]$$

$$= \frac{4(x^2 - 2xy + y^2)}{(x+y)^2}$$

$$\begin{aligned}
 &= \frac{4(x-y)^2}{(x+y)^2} \\
 &= \frac{4(x-y)^2(x+y)^2}{(x+y)^2(x+y)^2} \\
 &= \frac{4(x^2-y^2)^2}{(x+y)^2} \quad \text{---(4)}
 \end{aligned}$$

$$\therefore \left(\frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y} \right)^2 = 4 \left[1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} \right]$$

3. If $z = e^{ax+by} f(ax-by)$ show that

$$b \left[\frac{\partial z}{\partial x} \right] + a \left[\frac{\partial z}{\partial y} \right] = 2abz$$

$$\Rightarrow z = e^{ax+by} f(ax-by) \quad \text{---(1)}$$

$$\therefore \frac{\partial z}{\partial x} = e^{ax+by} (a) f(ax-by) + e^{ax+by} f(ax-by) (a)$$

$$b \frac{\partial z}{\partial x} = ab e^{ax+by} f(ax-by) + ab e^{ax+by} f(ax-by) \quad \text{---(2)}$$

Similarly,

$$\therefore \frac{\partial z}{\partial y} = e^{ax+by} b f(ax-by) + e^{ax+by} f(ax-by) (-b)$$

$$a \frac{\partial z}{\partial y} = ab e^{ax+by} f(ax-by) + e^{ax+by} f(ax-by) (-ab)$$

$$a \frac{\partial z}{\partial y} = ab e^{ax+by} f(ax-by) - ab e^{ax+by} f(ax-by) \quad \text{---(3)}$$

$$(2) + (3) \Rightarrow$$

$$\begin{aligned}
 b \frac{\partial z}{\partial x} + a \frac{\partial z}{\partial y} &= ab e^{ax+by} f(ax-by) + ab e^{ax+by} f(ax-by) + \\
 &\quad ab e^{ax+by} f(ax-by) - ab e^{ax+by} f(ax-by) \\
 &= 2ab e^{ax+by} f(ax-by) = 2abz
 \end{aligned}$$

4. If $u = f\left[\frac{x}{z}, \frac{y}{z}\right]$ then prove that

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = 0$$

$$\Rightarrow \text{Let, } p = \frac{x}{z}, \quad q = \frac{y}{z}$$

$$\text{then, } u = f[p, q]$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial x} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial x}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} \cdot \frac{1}{z} + \frac{\partial u}{\partial q} \cdot 0$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{1}{z} \frac{\partial u}{\partial p}$$

$$\Rightarrow x \frac{\partial u}{\partial x} = \frac{x}{z} \frac{\partial u}{\partial p} \quad \text{--- (1)}$$

$$\therefore \frac{\partial u}{\partial y} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial y} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial y}$$

$$\Rightarrow \frac{\partial u}{\partial y} = \frac{\partial u}{\partial p} (0) + \frac{\partial u}{\partial q} \left(\frac{1}{z}\right)$$

$$\Rightarrow y \frac{\partial u}{\partial y} = \frac{y}{z} \frac{\partial u}{\partial q} \quad \text{--- (2)}$$

$$\therefore \frac{\partial u}{\partial z} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial z} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial z}$$

$$\Rightarrow \frac{\partial u}{\partial z} = \frac{\partial u}{\partial p} \left(-\frac{x}{z^2}\right) + \frac{\partial u}{\partial q} \left(-\frac{y}{z^2}\right)$$

$$\Rightarrow z \frac{\partial u}{\partial z} = -\frac{x}{z^2} \frac{\partial u}{\partial p} - \frac{y}{z^2} \frac{\partial u}{\partial q} \quad \text{--- (3)}$$

$$(1) + (2) + (3)$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = \frac{x}{z} \frac{\partial u}{\partial p} + \frac{y}{z} \frac{\partial u}{\partial q} - \frac{x}{z^2} \frac{\partial u}{\partial p} - \frac{y}{z^2} \frac{\partial u}{\partial q}$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = 0$$

5. If $z = f(x+ay) + g(x-ay)$, prove that

$$\frac{\partial^2 z}{\partial y^2} = a^2 \left[\frac{\partial^2 z}{\partial x^2} \right]$$

$\Rightarrow$ Given,

$$z = f(x+ay) + g(x-ay) \text{ ——— (1)}$$

diff partially w.r to x

$$\therefore \frac{\partial z}{\partial x} = f'(x+ay)(1) + g'(x-ay)(1)$$

$$\Rightarrow \frac{\partial}{\partial x} \left[\frac{\partial z}{\partial x} \right] = \frac{\partial^2 z}{\partial x^2} = f''(x+ay)(1) + g''(x-ay)(1)$$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2} = f''(x+ay) + g''(x-ay) \text{ ——— (2)}$$

$$\therefore \frac{\partial z}{\partial y} = f'(x+ay)a + g'(x-ay)(-a)$$

$$\Rightarrow \frac{\partial^2 z}{\partial y^2} = f''(x+ay)a \cdot a + g''(x-ay)(-a)(-a)$$

$$\Rightarrow \frac{\partial^2 z}{\partial y^2} = a^2 f''(x+ay) + a^2 g''(x-ay)$$

$$\Rightarrow \frac{\partial^2 z}{\partial y^2} = a^2 [f''(x+ay) + g''(x-ay)]$$

$$\Rightarrow \frac{\partial^2 z}{\partial y^2} = a^2 \left[\frac{\partial^2 z}{\partial x^2} \right]$$

6. Find the extreme value of $f(x,y) = x^3 + 3x^2 + 4xy + y^2$

$\Rightarrow$ Given,

$$f(x,y) = x^3 + 3x^2 + 4xy + y^2 \text{ ——— (1)}$$

diff partially w.r to x

$$\frac{\partial f}{\partial x} = 3x^2 + 6x + 4y \text{ ——— (2)}$$

diff partially w.r to 'y'

$$\frac{\partial f}{\partial y} = 0 + 0 + 4x + 2y$$

$$\frac{\partial f}{\partial y} = 4x + 2y \quad \text{---(3)}$$

For max/min

$$\frac{\partial f}{\partial x} = 0$$

$$\frac{\partial f}{\partial y} = 0$$

$$(2) \Rightarrow 3x^2 + 6x + 4y \quad \text{---(4)}$$

$$(3) \Rightarrow 4x + 2y = 0$$

$$\Rightarrow 2y = -4x$$

$$\Rightarrow y = -2x \quad \text{---(5)}$$

$$(4) \Rightarrow 3x^2 + 6x + 4(-2x)$$

$$\Rightarrow 3x^2 + 6x - 8x$$

$$\Rightarrow 3x^2 - 2x$$

$$\Rightarrow x(3x - 2) = 0$$

$$\Rightarrow x = 0, \quad 3x - 2 = 0$$

$$\Rightarrow x = 0, \quad x = \frac{2}{3}$$

$$\Rightarrow y = 0, \quad y = -\frac{4}{3}$$

$$\therefore P(0, 0) \quad Q\left(\frac{2}{3}, -\frac{4}{3}\right)$$

$$\therefore \frac{\partial^2 f}{\partial x^2} = 6x + 6 = A$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left[\frac{\partial f}{\partial y} \right] = 4 = B$$

$$\frac{\partial^2 f}{\partial y^2} = 2 = C$$

At $P(0,0)$

$$A = 6(0) + 6 = 6$$

$$B = 4$$

$$C = 2$$

$$\begin{aligned}\therefore AC - B^2 &= 6(2) - 4^2 \\ &= 12 - 16 \\ &= -4 < 0\end{aligned}$$

$\therefore f(x,y)$ is neither max nor min at $P(0,0)$

At $Q\left(\frac{2}{3}, -\frac{4}{3}\right)$

$$A = 6\left[\frac{2}{3}\right] + 6 = 10 > 0$$

$$B = 4$$

$$C = 2$$

$$\begin{aligned}AC - B^2 &= (10)(2) - 4^2 \\ &= 20 - 16 \\ &= 4 > 0\end{aligned}$$

$\therefore f(x,y)$ is minima at $Q\left(\frac{2}{3}, -\frac{4}{3}\right)$

7. Examine the function $f(x,y) = xy(a-x-y)$ for extreme values:-

$\Rightarrow$ Given:-

$$f(x,y) = xy(a-x-y)$$

$$f(x,y) = axy - x^2y - xy^2 \quad \text{---(1)}$$

$$\therefore \frac{\partial f}{\partial x} = ay - 2xy - y^2 \quad \text{---(2)}$$

$$\frac{\partial f}{\partial y} = ax - x^2 - 2yx \quad \text{---(3)}$$

For max/min

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0$$

$$ay - 2xy - y^2 = 0$$

$$\Rightarrow y(a - 2x - y) = 0$$

$$\Rightarrow y = 0, \quad a - 2x - y = 0$$

$$\Rightarrow y = 0, \quad y = 2x - a$$

$$\therefore x = 0, y = 0$$

$$P(0,0)$$

$$x = \frac{3a}{2}, \quad y = 2\left[\frac{3a}{5}\right] - a$$
$$= \frac{6a - 5a}{5} = \frac{a}{5}$$

$$Q\left(\frac{3a}{5}, \frac{a}{5}\right)$$

$$\frac{\partial^2 f}{\partial x^2} = -2y = A$$

$$\frac{\partial^2 f}{\partial x \partial y} = a - 2x - 2y = B$$

$$\frac{\partial^2 f}{\partial y^2} = -2x = C$$

$$\therefore \text{At } P(0,0)$$

$$A = 0$$

$$B = a$$

$$C = 0$$

$$\therefore AC - B^2 = -a^2 < 0$$

$P(0,0)$ is saddle part.

$$ax - x^2 - 2yx = 0$$

$$\Rightarrow x(a - x - 2y) = 0$$

$$\Rightarrow x = 0, \quad a - x - 2y = 0$$

$$\Rightarrow x = 0, \quad a - x - 2(2x - a) = 0$$

$$\Rightarrow x = 0, \quad a - x - 4x + 2a = 0$$

$$\Rightarrow x = 0, \quad -5x + 3a = 0$$

$$\Rightarrow x = 0, \quad x = \frac{3a}{5}$$

$$\text{At } Q\left(\frac{3a}{5}, \frac{a}{5}\right)$$

$$A = -2\left[\frac{a}{5}\right] = -\frac{2a}{5} < 0$$

$$B = a - 2\left[\frac{3a}{5}\right] - \frac{2a}{5}$$

$$= a - \frac{6a}{5} - \frac{2a}{5}$$

$$= \frac{5a - 6a - 2a}{5}$$

$$= -\frac{3a}{5}$$

$$C = -2\left[\frac{3a}{5}\right]$$

$$= -\frac{6a}{5}$$

$$\therefore AC - B^2 = \left[-\frac{2a}{5}\right]\left[-\frac{6a}{5}\right] - \left[\frac{3a}{5}\right]^2$$

$$= \frac{12a^2}{25} - \frac{9a^2}{25}$$

$$= \frac{3a^2}{25} > 0$$

$\therefore f(x, y)$ is maximum at $Q\left(\frac{3a}{5}, \frac{a}{5}\right)$

$$\therefore \max f = a\left[\frac{3a}{5}\right]\left[\frac{a}{5}\right] - \left[\frac{3a}{5}\right]^2\left[\frac{a}{5}\right] - \left[\frac{3a}{5}\right]\left[\frac{a}{5}\right]^2$$

$$= \frac{3a^3}{25} - \frac{9a^3}{25} - \frac{3a^3}{25}$$

$$\max f = -\frac{9a^3}{25} \quad \forall \quad a > 0$$