Differential Equation of First Order and First Degree

An equation which contains one dependent variable and its derivates with respect to one or more variables is called differential equations.

There are 2 types :-

1. Ordinary Differential Equations [ODE]
2. Partial Differential Equations [PDE]

1. Ordinary Differential Equations [ODE]

An equation which contains one dependent variables and its derivatives with respect to one independent variables is called Ordinary Differential Equations [ODE].

Ex :-

$$\Rightarrow x^2 \frac{d^2y}{dx^2} - 2\left[\frac{dy}{dx}\right]^2 + y = 0$$

$$\Rightarrow x^2 \frac{d^2y}{dx^2} - 2xy\frac{dy}{dx} + 3y = 0$$

2. Partial Differential Equations [PDE]

An equation which contains one dependent variable and its derivatives with respect to two or more independent variables is called PDE.

Ex :-

$$\Rightarrow x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z$$

$$\Rightarrow x^2\frac{\partial^2 z}{\partial x^2} - 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial x} - 3\frac{\partial z}{\partial y} + z = 0$$

Solution of ODE of 1st Order and 1st Degree

Step 1 :- Write the given differential equation to the standard form.

$$M(x,y)\,dx + N(x,y)\,dy = 0 \quad \text{———} (1)$$

and identify M and N.

Step 2 :- Find $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$, if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then eqn (1) is called an exact differential equation and follows the solution as,

$$\int M(x,y)\,dx + \int (\text{The terms which do not contain } x \text{ in } N)\,dy = C$$

Step 03 :- If $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, the given D.E can be called as non-exact D.E

Step 04 :- Non-exact D.E may have a solutions by ~~reading~~ reducing to exact form as given below,

* Find $f(x)$ by $\frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right]$ and find the integrating factor $IF = e^{\int f(x)\,dx}$.
* Find $g(y)$ through $\frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] = g(y)$ and find the integrating factor $IF = e^{\int g(y)\,dy}$
* Multiply the integrating factor to the given D.E on both sides and verify the exactness.

Problems :-

1. Solve $\frac{dy}{dx} + \frac{2x+3y-1}{3x+4y-2} = 0$

Sol :- $\Rightarrow \frac{(3x+4y-2)\,dy + (2x+3y-1)\,dx}{(3x+4y-2)\,dx} = 0$

$$\Rightarrow (3x+4y-2)\,dy + (2x+3y-1)\,dx = 0$$

$$\Rightarrow (2x+3y-1)\,dx + (3x+4y-2)\,dy = 0 \quad \text{——— (1)}$$

$$M\,dx + N\,dy = 0$$

$\therefore M = 2x+3y-1 \quad , \quad N = 3x+4y-2$

$$\frac{\partial M}{\partial y} = 3 \quad , \quad \frac{\partial N}{\partial x} = 3$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Eqn (1) is an exact D.E

$\therefore$ The solution is ;

$$\int(2x+3y-1)\,dx + \int(4y-2)\,dy = C$$

$$\Rightarrow 2\int x\,dx + 3y\int 1\,dx - \int 1\,dx + 4\int y\,dy - 2\int 1\,dy = C$$

$$\Rightarrow \frac{\cancel{2}x^2}{\cancel{2}} + 3yx - x + \frac{{}^{2}\cancel{4}y^2}{\cancel{2}} - 2y = C$$

$$\Rightarrow x^2 + 3yx - x + 2y^2 - 2y = C$$

2. Solve $(2x+y+1)dx + (x+2y+1)dy = 0$

$$\Rightarrow (2x+y+1)\,dx + (x+2y+1)\,dy = 0 \quad \text{———(1)}$$

$$M\,dx + N\,dy = 0$$

$$M = 2x+y+1 \quad , \quad N = x+2y+1$$

$$\frac{\partial M}{\partial y} = 1 \quad , \quad \frac{\partial N}{\partial x} = 1$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Eqn (1) is an extact D.E

$\therefore$ The solution is :-

$$\int(2x+y+1)\,dx + \int(x+2y+1)\,dy = 0$$

$$\Rightarrow 2\int x\,dx + y\int 1\,dx + \int 1\,dx + 2\int y\,dy + \int 1\,dy = 0$$

$$\Rightarrow \cancel{2}\frac{x^2}{\cancel{2}} + yx + x + \frac{\cancel{2}y^2}{\cancel{2}} + y = C$$

$\Rightarrow x^2 + yx + y + x = C$

3. Solve $(5x^4+3x^2y^2-2xy^3)dx + (2x^3y-3x^2y^2-5y^4)dy = 0$

$\Rightarrow (5x^4+3x^2y^2-2xy^3)\,dx + (2x^3y-3x^2y^2-5y^4)\,dy = 0$

$$M\,dx + N\,dy = 0$$

$M = 5x^4+3x^2y^2-2xy^3$, $N = 2x^3y-3x^2y^2-5y^4$

$\frac{\partial M}{\partial y} = 6x^2y - 6xy^2$ $\qquad$ $\frac{\partial N}{\partial x} = 6x^2 - 6xy^2$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Eqn (1) is an EDE

$\therefore$ The Soln is ;

$\int(5x^4+3x^2y^2-2xy^3)dx + \int(-5y^4)dy = 0$

$\Rightarrow 5\int x^4 dx + 3y^2\int x^2 dx - 2y^3\int x\,dx - 5\int y^4 dy = C$

$\Rightarrow \cancel{5}\frac{x^5}{\cancel{5}} + \cancel{3}y^2\frac{x^3}{\cancel{3}} - \cancel{2}y^3\frac{x^2}{\cancel{2}} - \cancel{5}\frac{y^5}{\cancel{5}} = C$

$\Rightarrow x^5 + x^3y^2 - x^2y^3 - y^5 = C$

4. Solve, $\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$

$\Rightarrow (\sin x + x\cos y + x)\,dy + (y\cos x + \sin y + y)\,dx = 0$

$\Rightarrow (y\cos x + \sin y + y)\,dx + (\sin x + x\cos y + x)\,dy = 0$ —— (1)

$$M\,dx + N\,dy = 0$$

$M = y\cos x + \sin y + y$, $N = \sin x + x\cos y + x$

$\frac{\partial M}{\partial y} = \cos x + \cos y + 1$ $\qquad$ $\frac{\partial N}{\partial x} = \cos x + \cos y + 1$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Eqn (1) is an EDE

$\therefore$ The solution is :-

$$\int (y\cos x + \sin y + y)\,dx + \int 0 \cdot dy = c$$

$$\Rightarrow y\sin x + \sin y \int 1\,dx + y\int 1\,dx = c$$

$$\Rightarrow y\sin x + \sin y\, x + xy = c$$

5. Solve $(1+e^{x/y})\,dx + e^{x/y}(1 - x/y)\,dy = 0$

$$\Rightarrow (1+e^{x/y})\,dx + e^{x/y}(1 - x/y)\,dy = 0 \quad \text{———(1)}$$

$$M \cdot dx + N \cdot dy = 0$$

$M = 1 + e^{x/y}$

$$\frac{\partial M}{\partial y} = e^{x/y} \cdot -\frac{x}{y^2}$$

$$= -\frac{x}{y^2} e^{x/y}$$

$N = e^{x/y}(1 - x/y)$

$$\frac{\partial N}{\partial x} = e^{x/y}\left[-\frac{1}{y}\right] + \left[1 - \frac{x}{y}\right] e^{x/y} \cdot \frac{1}{y}$$

$$= \cancel{-\frac{e^{x/y}}{y}} + \cancel{\frac{e^{x/y}}{y}} - \frac{x}{y^2} e^{x/y}$$

$$= -\frac{x}{y^2} e^{x/y}$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Eqn (1) is an exact d.e

$$\therefore \int (1+e^{x/y})\,dx + \int 0 \cdot dy = c$$

$$\Rightarrow x + \frac{e^{x/y}}{1/y} = c$$

$$\Rightarrow x + y e^{x/y} = c$$

6. Solve $[4x^3y^2 + y\cos(xy)]dx + [2x^4y + x\cos(xy)]dy$

$\Rightarrow [4x^3y^2 + y\cos(xy)]\,dx + [2x^4y + x\cos(xy)]\,dy = 0$ —(1)

$$M.dx + N.dy = 0$$

$M = 4x^3y^2 + y\cos(xy)$

$\Rightarrow \dfrac{\partial M}{\partial y} = 8x^3y + y(-\sin xy)x + \cos(xy)(1)$

$= 8x^3y + xy(-\sin xy) + \cos xy$

$= 8x^3y - xy\sin xy + \cos xy$

$N = 2x^4y + x\cos(xy)$

$\Rightarrow \dfrac{\partial N}{\partial x} = 8x^3y + x(-\sin xy)y + \cos(xy)(1)$

$= 8x^3y - xy\sin xy + \cos xy$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Eqn (1) is an EDE

$\therefore$ The solution is,

$\int [4x^3y^2 + y\cos(xy)]dx + \int 0\,dy = c$

$\Rightarrow \dfrac{\cancel{4}x^4y^3}{\cancel{4}} + \dfrac{\cancel{y}\sin(xy)}{\cancel{y}} = c$

$\Rightarrow \quad x^4y^3 + \sin(xy) = c$

7. Solve $\left[y\left(1+\frac{1}{x}\right) + \cos y\right]dx + [x + \log x - x\sin y]\,dy = 0$

$\Rightarrow \left[y\left[1+\frac{1}{x}\right] + \cos y\right]dx + [x + \log x - x\sin y]\,dy = 0$

$$M\,dx + N\,dy = 0$$

$M = y(1+\frac{1}{x}) + \cos y$

$M = y + \frac{y}{x} + \cos y$

$N = x + \log x - x\sin y$

$\frac{\partial N}{\partial x} = 1 + \frac{1}{x} - \sin y$

$\frac{\partial M}{\partial y} = 1 + \frac{1}{x} - \sin y$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Eqn (1) is an exact d.e

$\therefore$ The soln is,

$\int (y(1+\frac{1}{x}) + \cos y)\, dx + \int 0\, dy = c$

$\Rightarrow y\int 1 \cdot dx + y\int \frac{1}{x} dx + \cos y \int 1 \cdot dx = c$

$\Rightarrow xy + y\log x + x\cos y = c$

8. Solve $(x^2+y^2+x)dx + xy\, dy = 0$

$\Rightarrow (x^2+y^2+x)\, dx + xy\, dy = 0$

$M\,dx + N\,dy$

$M = x^2+y^2+x$

$N = xy$

$\frac{\partial M}{\partial y} = 2y$

$\frac{\partial N}{\partial x} = y$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

let,

$$\frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = \frac{1}{xy}(2y-y)$$

$$= \frac{y}{xy} = \frac{1}{x} = f(x)$$

$\therefore$ IF $= e^{\int f(x)dx}$

$= e^{\int \frac{1}{x} dx}$

$= e^{\log_e x}$

$= x$

1) $\Rightarrow$ Eqn① $\times$ IF $\Rightarrow x(x^2+y^2+x)dx + x \cdot xy\, dy = 0$

$\Rightarrow (x^3+xy^2+x^2)dx + x^2y\, dy = 0$ —(2)

$\Rightarrow M'dx + N'dy = 0$

$\therefore \frac{\partial M'}{\partial y} = 2xy$, $\frac{\partial N'}{\partial x} = 2xy$

$\therefore \frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x}$

eqn (2) is an EDE

$\therefore$ The solution is,

$\int [x^3+xy^2+x^2]\, dx + \int 0\, dy = c$

$\Rightarrow \int x^3 dx + y^2 \int x\, dx + \int x^2 dx = c$

$\Rightarrow \frac{x^4}{4} + y^2 \frac{x^2}{2} + \frac{x^3}{3} = c$

$\Rightarrow 3x^4 + 6x^2y^2 + 4x^3 = 12c$

9. $(4xy + 3y^2 - x)dx + x(x+2y)\, dy = 0$

$\Rightarrow (4xy+3y^2-x)dx + x(x+2y)\, dy = 0$ —(1)

$M dx + N dy = 0$

$M = 4xy+3y^2-x$ $\quad$ $N = x(x+2y) = x^2+2xy$

$\frac{\partial M}{\partial y} = 4x+6y$ $\quad$ $\frac{\partial N}{\partial x} = 2x+2y$

Let,

$$\frac{1}{N}\left[\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right]=\frac{1}{x(x+2y)}\,[2x+4y]$$

$$=\frac{2\cancel{(x+2y)}}{x\cancel{(x+2y)}}$$

$$=\frac{2}{x}=f(x)$$

$$\text{I.F}=e^{\int f(x)dx}$$

$$=e^{\int \frac{2}{x}dx}$$

$$=e^{2\int \frac{1}{x}dx}$$

$$=e^{2\log_e x}$$

$$=e^{\log_e x^2}$$

$$=x^2$$

$$① \times \text{IF} \Rightarrow x^2(4xy+3y^2-x)\,dx+x^2.x(x+2y)\,dy=0$$

$$\Rightarrow (4x^3y+3x^2y^2-x^3)\,dx+(x^4+2x^3y)\,dy=0$$

$$\frac{\partial M'}{\partial y}=\frac{\partial N'}{\partial x}$$

eqn (2) is EDE

$\therefore$ The solution is,

$$\int(4x^3y+3y^2x^2-x^3)\,dx-\int 0\,dy=C$$

$$\Rightarrow 4y\int x^3dx+3y^2\int x^2\,dx-\int x^3dx=C$$

$$\Rightarrow \frac{\cancel{4}yx^4}{\cancel{4}}+\frac{\cancel{3}y^2x^3}{\cancel{3}}-\frac{x^4}{4}=C$$

$$\Rightarrow yx^4+y^2x^3-\frac{x^4}{4}=C$$

10. Solve $(x^3+y^3+6x)\,dx + (xy)^2\,dy = 0$

$\Rightarrow \quad (x^3+y^3+6x)\,dx + (xy)^2\,dy = 0$ ——(1)

$$M\,dx + N\,dy = 0$$

$M = x^3+y^3+6x \qquad\qquad N = xy^2$

$\dfrac{\partial M}{\partial y} = 3y^2 \qquad\qquad \dfrac{\partial N}{\partial x} = y^2$

let, $\dfrac{1}{N}\left[\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right] = \dfrac{1}{xy^2}\left[3y^2-y^2\right]$

$= \dfrac{1}{xy^2}\left[2y^2\right]$

$= \dfrac{2}{x} = f(x)$

$I.F = e^{\int f(x)dx}$

$= e^{\int 2/x\,dx}$

$= e^{2\int 1/x\,dx}$

$= e^{2\log x}$

$= e^{\log_e x^2}$

$= x^2$

(1) $\times$ IF $= x^2\left[x^3+y^3+6x\right]dx + x^2\left[xy^2\right]dy = 0$

$= x^5 + x^2y^3 + 6x^3\,dx + x^3y^2\,dy = 0$ ——(2)

$\therefore \dfrac{\partial M'}{\partial y} = \dfrac{\partial N'}{\partial x}$

eqn (2) is an EDE

The solution is,

$\int x^5 + x^2y^3 + 6x^3\,dx + \int 0\,dy = c$

$\Rightarrow \int x^5 dx + y^3\int x^2 dx + 6\int x^3 dx = C$

$$\Rightarrow \frac{x^6}{6} + y^3\frac{x^3}{3} + 6\frac{x^4}{4} = c$$

$$\Rightarrow 2x^6 + 4x^3y^3 + 18x^4 = 12c$$

11. Solve $(y\log y)dx + (x-\log y)dy = 0$

$$\Rightarrow (y\log y)dx + (x-\log y)dy = 0 \quad \text{——(1)}$$

$$Mdx + Ndy = 0$$

$M = y\log y$ $\qquad\qquad$ $N = x - \log y$

$$\frac{\partial M}{\partial y} = 1.\log y + y\cdot\frac{1}{y} \qquad \frac{\partial N}{\partial x} = 1$$

$$= \log y + 1$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Let, $\frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] = \frac{1}{y\log y}\left[1-(\log y+1)\right]$

$$= \frac{1}{y\log y}\left[\cancel{1} - \log y - \cancel{1}\right]$$

$$= -\frac{\cancel{\log y}}{y\cancel{\log y}}$$

$$= -\frac{1}{y} = g(y)$$

$$\therefore IF = e^{\int g(y)dy}$$

$$= e^{\int -\frac{1}{y}dy}$$

$$= e^{-\log y}$$

$$= e^{\log(y^{-1})}$$

$$= e^{\log(1/y)}$$

$$= \frac{1}{y}$$

$$① \times I.F \Rightarrow \frac{1}{y}[y\log y]dx + \frac{1}{y}[x - \log y]dy = 0$$

$$\Rightarrow \log y\, dx + \left[\frac{x}{y} - \frac{\log y}{y}\right] dy = 0$$

$$\Rightarrow \log y\, dx + \left[\frac{x}{y} - \frac{\log y}{y}\right] dy = 0$$

$$\frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x}$$

The solution is,

$$\int \log y\, dx + \int \left[-\frac{\log y}{y}\right] dy = c$$

$$\Rightarrow \log y \int 1 dx + \int \frac{1}{y} \log y\, dy = c \quad \text{——(3)}$$

Let, $I = \int \frac{1}{y} \log y$

formula :- $\int uv\,dx = u\int v\,dx - \int(u' \int v\,dx)\,dx$

$$\Rightarrow I = \log y \cdot \int \frac{1}{y} dx - \int \left(\frac{1}{y} \int \frac{1}{y} dy\right) dx$$

$$\Rightarrow I = \log y \cdot \log y - \int \frac{1}{y} \log y\, dy$$

$$\Rightarrow I = (\log y)^2 - I$$

$$\Rightarrow I + I = (\log y)^2$$

$$\Rightarrow 2I = (\log y)^2$$

$$\Rightarrow I = \frac{1}{2}(\log y)^2$$

from eqn (3)

$$\Rightarrow x \log y - \frac{1}{2}(\log y)^2 = c$$

12. Solve $y(2xy+1)dx - x\,dy = 0$

$\Rightarrow \quad y(2xy+1)\,dx - x\,dy = 0 \quad \text{——} (1)$

$$M\,dx + N\,dy = 0$$

$M = y(2xy+1)$ $\qquad\qquad$ $N = -x$

$M = 2xy^2 + y$ $\qquad\qquad$ $\dfrac{\partial N}{\partial x} = -1$

$\dfrac{\partial M}{\partial y} = 4xy + 1$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

let, $\dfrac{1}{M}\left[\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}\right] = \dfrac{1}{y(2xy+1)}[-1-4xy-1]$

$$= \frac{-2-4xy}{y(2xy+1)}$$

$$= \frac{-2(1+\cancel{2xy})}{y(1+\cancel{2xy})}$$

$$= -\frac{2}{y} = g(y)$$

$$\text{I.F} = e^{\int g(y)\,dy}$$

$$= e^{-2\int \frac{1}{y}\,dy}$$

$$= e^{-2\log y}$$

$$= e^{\log_e(1/y^2)}$$

$$= \frac{1}{y^2}$$

$(1) \times \text{I.F} \Rightarrow \dfrac{1}{y^{\cancel{2}}}\,\cancel{y}(1+2xy)\,dx - \dfrac{x}{y^2}\,dy = 0$

$\Rightarrow \dfrac{1}{y}(1+2xy)\,dx - \dfrac{x}{y^2}\,dy = 0$

$$\Rightarrow \left[\frac{1}{y} + 2x\right] dx - \frac{x}{y^2} dy = 0$$

$$\frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x}$$

$\therefore$ The solution is,

$$\Rightarrow \int\left(\frac{1}{y} + 2x\right) dx + \int 0\, dy = c$$

$$\Rightarrow \frac{1}{y}\int 1\, dx + 2\int x\, dx = c$$

$$\Rightarrow \frac{x}{y} + x^2 = c$$

Solution of Linear Differential Equation of First Order and First Degree.

Step 01 :- Write the given differential equation to the standard form.

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Step 02 :- Identify P and Q and find the integrating factor

$$IF = e^{\int P(x)\, dx}$$

Step 03 :- Write the soln as given below

$$y \times IF = \int Q(x) \times IF\, dx + c$$

Similarly, for the D.E

$$\frac{dx}{dy} + P(y)x = Q(y)$$

we get the soln

$$x \times IF = \int Q(y) \times IF\, dy + c$$

when, $IF = e^{\int P(y)\, dy}$

Bernoulli's Equation

Step 01 :- Define the Bernoulli's equation

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \quad \text{——} (1)$$

Step 02 :- Divide y^n on b.s and reduce the equation as,

$$\text{eqn}(1) \Rightarrow \frac{1}{y^n}\frac{dy}{dx} + P(x) \cdot \frac{1}{y^{n-1}} = Q(x) \quad \text{——} (2)$$

Step 03 :- Assume

$$\frac{1}{y^{n-1}} = u$$ and differentiate the eqn w.r to 'x'

$$\Rightarrow \frac{1}{y^n}\frac{dy}{dx} = \frac{1}{1-n}\frac{dy}{dx}$$

Step 04 :- Reduce the eqn (2) to the linear form and follow the same procedure.

Step 05 :- The similar process can be applicable for the Bernoulli's equation.

$$\frac{dx}{dy} + P(y)x = Q(y)x^n$$

1. Solve $\frac{dy}{dx} - \frac{y}{x} = 2x^2$

$$\Rightarrow \frac{dy}{dx} - \frac{y}{x} = 2x^2$$

$$\Rightarrow \frac{dy}{dx} + \left[-\frac{1}{x}\right]y = 2x^2$$

where, $P = -\frac{1}{x}$ $\quad Q = 2x^2$

$\therefore$ I.F $= e^{\int P(x)\,dx}$

$= e^{\int -\frac{1}{x}\,dx}$

$= e^{-\log x}$

$= e^{\log_e(1/x)}$

$= \frac{1}{x}$

The solution is,

$y \times IF = \int Q \times IF\,dx + c$

$\Rightarrow y\left[\frac{1}{x}\right] = \int 2x^2 \cdot \frac{1}{x}\,dx + c$

$\Rightarrow \frac{y}{x} = 2\int x\,dx + c$

$\Rightarrow \frac{y}{x} = \cancel{2}\frac{x^2}{\cancel{2}} + c$

$\Rightarrow \frac{y}{x} = x^2 + c$

2. Solve, $\frac{dy}{dx} + \frac{y}{x} = y^2 x$

$\Rightarrow \frac{dy}{dx} + \frac{y}{x} = y^2 x$ ——(1)

$\Rightarrow \frac{dy}{dx} + \frac{1}{x}(y) = y^2 x$

$\Rightarrow \frac{1}{y^2}\frac{dy}{dx} + \frac{1}{x}\frac{1}{y} = \frac{x\cancel{y^2}}{\cancel{y^2}}$

$\Rightarrow \frac{1}{y^2}\frac{dy}{dx} + \frac{1}{x}\cdot\frac{1}{y} = x$ ——(2)

Let, $\frac{1}{y^2} = u$ ——(3)

diff (3) w.r to x

$$(3) \Rightarrow \frac{1}{y^2}\frac{dy}{dx} = \frac{du}{dx}$$

$$\Rightarrow \frac{1}{y^2}\frac{dy}{dx} = -\frac{du}{dx}$$

$$(2) \Rightarrow -\frac{du}{dx} + \frac{1}{x}u = x$$

$$\Rightarrow \frac{du}{dx} - \frac{1}{x}u = -x$$

$$\Rightarrow \frac{du}{dx} + \left[-\frac{1}{x}\right]u = -x$$

$$\therefore P = -\frac{1}{x}, \quad Q = -x$$

$$I.F = e^{-\int 1/x \, dx}$$

$$= e^{-\log x}$$

$$= e^{\log(1/x)}$$

$$= \frac{1}{x}$$

The solution is,

$$u \times IF = \int Q(x) \times IF \, dx + C$$

$$\Rightarrow u\left(\frac{1}{x}\right) = -\int x \cdot \frac{1}{x} \, dx + C$$

$$\Rightarrow \frac{u}{x} = -\int 1 \, dx + C$$

$$\Rightarrow \frac{u}{x} = -x + C$$

$$\Rightarrow \frac{1}{xy} = -x + C$$

$$\Rightarrow \frac{1}{xy} + x = C$$

3. Solve, $\frac{dy}{dx} + xy = xy^3$

$\Rightarrow \quad \frac{dy}{dx} + xy = xy^3$ ——— (1)

$\Rightarrow \quad \frac{1}{y^3}\frac{dy}{dx} + \frac{xy}{y^3} = \frac{x\cancel{y^3}}{\cancel{y^3}}$

$\Rightarrow \quad \frac{1}{y^3}\frac{dy}{dx} + \frac{x}{y^2} = x$ ——— (2)

Let, $\frac{1}{y^2} = u$ ——— (3)

diff w.r to 'x'

(3) $\Rightarrow -\frac{2}{y^3}\frac{dy}{dx} = \frac{du}{dx}$

$\Rightarrow \frac{1}{y^3}\frac{dy}{dx} = -\frac{1}{2}\frac{du}{dx}$

(2) $\Rightarrow -\frac{1}{2}\frac{du}{dx} + xu = x$

$\Rightarrow \frac{du}{dx} - 2xu = -2x$ $(\div(2))$

$\Rightarrow \frac{du}{dx} + (-2x)u = -2x$

$p = -2x \quad , \quad Q = -2x$

$\therefore \ IF = e^{\int p(x)dx}$

$= e^{-2\int x\,dx}$

$= e^{-2\frac{x^2}{2}}$

$= e^{-x^2}$

$\therefore$ The solution is,

$u \times IF = \int Q(x) \times I.F\, dx + c$

$\Rightarrow \ ue^{-x^2} = \int (-2x)e^{-x^2} dx + c$

$$\Rightarrow \frac{e^{-x^2}}{y^2} = \int e^{-x^2}(-2x)\,dx + c \quad \text{——— (4)}$$

Let, $-x^2 = t$

$$\Rightarrow -2x\,dx = dt$$

$$\therefore (4) \Rightarrow \frac{e^{-x^2}}{y^2} = \int e^t\,dt + c$$

$$\Rightarrow \frac{e^{-x^2}}{y^2} = e^t + c$$

$$\Rightarrow \frac{e^{-x^2}}{y^2} = e^{-x^2} + c$$

$$\Rightarrow e^{-x^2} = (e^{-x^2} + c)\,y^2$$

4. Solve $x\left[\frac{dy}{dx}\right] + y = x^3y^6$

$$\Rightarrow x\left[\frac{dy}{dx}\right] + y = x^3y^6$$

$$\Rightarrow \frac{dy}{dx} + \frac{1}{x}y = x^2y^6 \quad \text{——— (1)}$$

$$\Rightarrow \frac{1}{y^6}\frac{dy}{dx} + \left[\frac{1}{x}\right]\left[\frac{1}{y^5}\right] = x^2 \quad \text{——— (2)}$$

let, $\frac{1}{y^5} = u$

$$\Rightarrow -\frac{5}{y^6}\cdot\frac{dy}{dx} = \frac{du}{dx}$$

$$\Rightarrow \frac{1}{y^6}\frac{dy}{dx} = -\frac{1}{5}\frac{du}{dx}$$

$$(2) \Rightarrow -\frac{1}{5}\frac{du}{dx} + \frac{1}{x}u = x^2$$

$$\Rightarrow \frac{du}{dx} + \left[-\frac{5}{x}\right]u = -5x^2 \quad \text{---} (3)$$

$\therefore \; P = -\frac{5}{x}, \quad Q = -5x^2$

$$I.F = e^{\int P(x)dx}$$

$$= e^{-5\int \frac{1}{x}dx}$$

$$= e^{-5\log x}$$

$$= e^{\log_e(1/x^5)}$$

$$= \frac{1}{x^5}$$

$\therefore$ The solution is,

$$u \times I.F = \int Q \times I.F\, dx + c$$

$$\Rightarrow u\left(\frac{1}{x^5}\right) = -\int 5x^2 \cdot \frac{1}{x^5}\, dx + c$$

$$\Rightarrow \frac{u}{x^5} = -5\int \frac{1}{x^3}\, dx + c$$

$$\Rightarrow \frac{u}{x^5} = -5\, \frac{x^{-3+1}}{-3+1} + c$$

$$\Rightarrow \frac{u}{x^5} = \frac{-5}{-2} x^{-2} + c$$

$$\Rightarrow \frac{1}{x^5 y^5} = \frac{5}{2} x^2 + c$$

5. Solve, $\frac{dy}{dx} - y\tan x = y^2 \sec x$

$\Rightarrow \quad \frac{dy}{dx} - y\tan x = y^2 \sec x$

$\Rightarrow \quad \frac{1}{y^2}\frac{dy}{dx} - (\tan x)\left(\frac{1}{y}\right) = \sec x \quad \text{——} (1)$

let, $\frac{1}{y} = u$

$\Rightarrow \quad -\frac{1}{y^2}\frac{dy}{dx} = \frac{du}{dx}$

$\Rightarrow \quad \frac{1}{y^2}\frac{dy}{dx} = -\frac{du}{dx}$

$(1) \Rightarrow \quad -\frac{du}{dx} - (\tan x)u = \sec x$

$\Rightarrow \quad \frac{du}{dx} + \tan x \cdot u = -\sec x \quad \text{——} (2)$

$p = \tan x \quad , Q = -\sec x$

$\therefore \quad I.F = e^{\int \tan x \, dx}$

$= e^{\log_e |\sec x|}$

$= \sec x$

$\therefore$ The solution is,

$u \sec x = -\int \sec x \cdot \sec x \, dx + c$

$\Rightarrow \frac{1}{y}\sec x = -\int \sec^2 x \, dx + c$

$\Rightarrow \frac{1}{y}\sec x = -\tan x + c$

$\Rightarrow \frac{1}{y}\sec x + \tan x = c$

6. Solve, $\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y$

$\Rightarrow \quad \frac{dy}{dx} + x\sin 2y = x^3\cos^2 y$

$\Rightarrow \quad \frac{1}{\cos^2 y}\frac{dy}{dx} + 2x\frac{\sin y\cos y}{\cos^2 y} = x^3$

$\Rightarrow \quad \sec^2 y\frac{dy}{dx} + 2x\frac{\sin y}{\cos y} = x^3$

$\Rightarrow \quad \sec^2 y\frac{dy}{dx} + 2x\tan y = x^3 \text{ ——(1)}$

let, $\tan y = u$

$\sec^2 y \cdot \frac{dy}{dx} = \frac{du}{dx}$

$\Rightarrow \quad \frac{du}{dx} + 2xu = x^3$

$\therefore P = 2x, \quad Q = x^3$

$\therefore IF = e^{2\int x\,dx}$

$= e^{x^2}$

The solution is;

$ue^{x^2} = \int x^3 e^{x^2}dx + c$

$\Rightarrow \quad e^{x^2}\tan y = \int x^2 \cdot e^{x^2} \cdot x\,dx + c \text{ ——(2)}$

let, $x^2 = t$

$2x\,dx = dt$

$x\,dx = \frac{1}{2}dt$

$(2) \Rightarrow \quad e^{x^2}\tan y = \frac{1}{2}\int t \cdot e^t dt + c$

$$\Rightarrow e^{x^2} \tan y = \frac{1}{2} e^{t}(t-1) + c$$

$$\Rightarrow e^{x^2} \tan y = \frac{1}{2} e^{x^2}(x^2-1) + c$$

Solve $xy(1+xy^2)\frac{dy}{dx} = 1$

$\Rightarrow$ Given,

$$\Rightarrow xy(1+xy^2)\frac{dy}{dx} = 1$$

$$\Rightarrow (xy + x^2y^3)\frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{xy + x^2y^3}$$

$$\Rightarrow \frac{dx}{dy} = xy + x^2y^3$$

$$\Rightarrow \frac{dx}{dy} - xy = x^2y^3$$

Let, $\frac{1}{x} = u$

$$\Rightarrow -\frac{1}{x^2}\frac{dx}{dy} = \frac{du}{dy}$$

$$\Rightarrow \frac{1}{x^2}\frac{dx}{dy} = -\frac{du}{dy}$$

$$\therefore (1) \Rightarrow -\frac{du}{dy} - yu = y^3$$

$$\Rightarrow \frac{du}{dy} + yu = -y^3$$

$$\therefore P(y) = y \quad , \quad Q(y) = -y^3$$

$$I.F = e^{\int P(y)\,dy}$$
$$= e^{\int y\,dy}$$
$$= e^{y^2/2}$$

$\therefore$ The solution is,

$$u e^{y^2/2} = -\int y^3 . e^{y^2/2} dy + c$$

$$\Rightarrow \frac{1}{x} e^{y^2/2} = -\int y^2 . e^{y^2/2} . y\,dy + c \quad \text{---(2)}$$

Let, $\frac{y^2}{2} = t$

$$\Rightarrow y^2 = 2t$$
$$\Rightarrow 2y\,dy = 2\,dt$$
$$\Rightarrow y\,dy = dt$$

$$\therefore (2) \Rightarrow \frac{1}{x} e^{y^2/2} = -\int y^2 . e^t\,dt + c$$
$$\Rightarrow \frac{1}{x} e^{y^2/2} = -\frac{y^3}{3} e^t + c$$

Orthogonal Trajactories :-

If the given two curves can intersect perpendicularly then those curves are said to be orthogonally to each other.

Orthogonal Trajactory In Cartesian Form :-

Step 1 :- Let the given curve be $F(x, y, c) = 0$ —(1)

Step 2 :- Diff eqn (1) w.r to 'x' and get the differential equation as

$$f\left[x, y, \frac{dy}{dx}\right] = 0 \quad \text{——— (2)}$$

Step 2 :- To get orthogonally trajactory, replace $\frac{dy}{dx} = -\frac{dx}{dy}$ in eqn (2), we get

$$(2) \Rightarrow f\left[x, y, -\frac{dx}{dy}\right] = 0 \quad \text{——— (3)}$$

Step 4 :- Solve eqn (3) and get the solution, say $g(x, y, c') = 0$, which is called the orthogonal trajectory of the given curve.

Orthogonal Trajactory In Polar Form :-

Step 1 :- Let the given polar curve be

$$F(r, \theta, c) = 0 \quad \text{——— (1)}$$

Step 2 :- Diff eqn (1) w.r.to 'θ' and get the

$$F\left(r, \theta, \frac{dr}{d\theta}\right) = 0 \quad \text{——— (2)}$$

Step 3 :- To get orthogonally trajectory, replace $\frac{dr}{d\theta} = -r^2 \frac{d\theta}{dr}$, we get

$$(2) \Rightarrow f\left(r, \theta, -r^2 \frac{d\theta}{dr}\right) = 0 \quad \text{——— (3)}$$

Step 4 :- Solve eqn (3) and get the solution $g(r, \theta, c') = 0$ which is called the orthogonal trajectory of the polar curves.

Problems :-

1. Find the orthogonally trajectory of the parabola $y^2 = 4ax$, where 'a' is the parameter.

$\Rightarrow$ Given,

$$y^2 = 4ax \quad \text{——(1)}$$

diff (1) w.r to 'x'

$$\therefore (1) \Rightarrow 2y\frac{dy}{dx} = 4a$$

$$\Rightarrow y^2 = 2y\frac{dy}{dx}\cdot x$$

$$\Rightarrow y = 2x\frac{dy}{dx} \quad \text{——(2)}$$

replace, $\frac{dy}{dx} = -\frac{dx}{dy}$

$$\therefore (2) \Rightarrow y = 2x\left[-\frac{dx}{dy}\right]$$

$$\Rightarrow y = -2x\frac{dx}{dy}$$

$$\Rightarrow y\,dy = -2x\cdot dx$$

$$\Rightarrow \int y\,dy = -2\int x\,dx$$

$$\Rightarrow \frac{y^2}{2} = -\cancel{2}\frac{x^2}{\cancel{2}} + c$$

$$\Rightarrow x^2 + \frac{y^2}{2} = c$$

2). Show that the parabola $y^2 = 4a(x+a)$ is self-orthogonal.

$\Rightarrow$ $y^2 = 4a(x+a)$ —— (1)

diff (1) w.r to 'x'

(1) $\Rightarrow$ $2yy_1 = 4a(1+0)$

$\Rightarrow$ $yy_1 = 2a$

$\Rightarrow$ $a = \frac{yy_1}{2}$

$\therefore$ (1) $\Rightarrow$ $y^2 = 4\left[\frac{yy_1}{2}\right]\left[x + \frac{yy_1}{2}\right]$

$\Rightarrow$ $y^2 = 4\,\frac{yy_1}{2}\left[\frac{2x + yy_1}{2}\right]$

$\Rightarrow$ $y^2 = yy_1(2x + yy_1)$

$\Rightarrow$ $y = y_1(2x + yy_1)$ —— (2) Replacing, $y_1 = -\frac{1}{y_1}$

(2) $\Rightarrow$ $y = 2x\left[-\frac{1}{y_1}\right] + y\left[-\frac{1}{y_1}\right]^2$

$\Rightarrow$ $y = -\frac{2x}{y_1} + \frac{y}{y_1^2}$

$\Rightarrow$ $y = \frac{-2xy_1 + y}{y_1^2}$

$\Rightarrow$ $yy_1^2 = -2xy_1 + y$

$\Rightarrow$ $y = 2xy_1 + yy_1^2$ —— (3)

Eqn (2) & (3) are equal

$\therefore$ Eq^n is self orthogonally.

3. Find the orthogonal trajactory of $\frac{x^2}{a^2}+\frac{y^2}{a^2+\lambda}=1$ where 'λ' is the parameter.

$\Rightarrow$ Given : $\frac{x^2}{a^2}+\frac{y^2}{a^2+\lambda}=1$ ——(1)

diff (1) w.r to 'x'

$$\therefore (1) \Rightarrow \frac{2x}{a^2}+\frac{2yy_1}{a^2+\lambda}=0$$

$$\Rightarrow \frac{x}{a^2}+\frac{yy_1}{a^2+\lambda}=0$$

$$\Rightarrow \frac{yy_1}{a^2+\lambda}=-\frac{x}{a^2}$$

$$\Rightarrow \frac{1}{a^2+\lambda}=\frac{-x}{a^2yy_1}$$

$$\Rightarrow a^2+\lambda=\frac{a^2yy_1}{-x}$$

$$\therefore (1) \Rightarrow \frac{x^2}{a^2}+\frac{y^2}{\frac{-a^2yy_1}{x}}=1$$

$$\Rightarrow x^2-\frac{y}{y_1/x}=a^2$$

$$\Rightarrow x^2-\frac{xy}{y_1}=a^2 \quad \text{——(2)}$$

let, $y_1=-\frac{1}{y_1}$

$$\therefore (2) \Rightarrow x^2-\frac{xy}{(-1/y_1)}=a^2$$

$$\Rightarrow x^2-xyy_1=a^2$$

$$\Rightarrow \quad yy_1 = \frac{a^2 - x^2}{x}$$

$$\Rightarrow \quad y\frac{dy}{dx} = \frac{a^2}{x} - \frac{x^2}{x}$$

$$\Rightarrow \quad y\,dy = \left[\frac{a^2}{x} - x\right] dx$$

$$\Rightarrow \quad \int y \cdot dy = a^2 \int \frac{1}{x} dx - \int x \cdot dx$$

$$\Rightarrow \quad \frac{y^2}{2} = a^2 \log x - \frac{x^2}{2} + c$$

$$\Rightarrow \quad \frac{x^2}{2} + \frac{y^2}{2} = a^2 \log x + c$$

4. Show that $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$ is self orthogonal where 'λ' is the parameter.

$\Rightarrow$ Given :- $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$ ——— (1)

diff (1) w. r to 'x'

$$\therefore (1) \Rightarrow \frac{2x}{a^2+\lambda} + \frac{2yy_1}{b^2+\lambda} = 0$$

$$\Rightarrow \frac{x}{a^2+\lambda} + \frac{yy_1}{b^2+\lambda} = 0$$

$$\Rightarrow (b^2+\lambda)x + (a^2+\lambda)yy_1 = 0$$

$$\Rightarrow b^2x + \lambda x + a^2yy_1 + \lambda yy_1 = 0$$

$$\Rightarrow (x + yy_1)\lambda = -(b^2x) + (a^2yy_1)$$

$$\Rightarrow \lambda = -\frac{(b^2x + a^2yy_1)}{x + yy_1}$$

$$\therefore\ a^2+\lambda = a^2 - \frac{(b^2x+a^2yy_1)}{x+yy_1}$$

$$= \frac{a^2(x+yy_1)-(b^2x+a^2yy_1)}{x+yy_1}$$

$$= \frac{a^2x + a^2yy_1 - b^2x - a^2yy_1}{x+yy_1}$$

$$= \frac{a^2x + a^2\cancel{yy_1} - b^2x - a^2\cancel{yy_1}}{x+yy_1}$$

$$= \frac{(a^2-b^2)x}{x+yy_1}$$

$$\therefore\ b^2+\lambda = b^2 - \frac{(b^2x+a^2yy_1)}{x+yy_1}$$

$$= \frac{b^2x + b^2yy_1 - b^2x - a^2yy_1}{x+yy_1}$$

$$= -\frac{(a^2-b^2)yy_1}{x+yy_1}$$

$$\therefore (1) \Rightarrow \frac{x^2}{\frac{(a^2-b^2)x}{x+yy_1}} + \frac{y^2}{-\frac{(a^2-b^2)yy_1}{x+yy_1}} = 1$$

$$\Rightarrow \frac{x(x+yy_1)}{1} - \frac{y(x+yy_1)}{y_1} = a^2-b^2$$

$$\Rightarrow (x+yy_1)\left(x-\frac{y}{y_1}\right) = a^2-b^2 \quad \text{---}(2)$$

Let , $y_1 = -\frac{1}{y_1}$

$$\therefore (2) \Rightarrow \left[x+y\left[-\frac{1}{y_1}\right]\right]\left[x-\frac{y}{\left(-1/y_1\right)}\right] = a^2-b^2$$

$$\Rightarrow \left[x-\frac{y}{y_1}\right]\left[x+yy_1\right] = a^2-b^2 \quad \text{——(3)}$$

$\therefore$ Eqn (2) & (3) are equal

$\therefore$ The eqn is self orthogonal.

5. Find the orthogonal trajactory of the given polar curve.

i) $r = a(1-\cos\theta)$

$\Rightarrow$ Given : $r = a(1-\cos\theta)$ ——(1)

diff . w.r to 'θ'

$$\therefore (1) \Rightarrow \frac{dr}{d\theta} = a\,(0+\sin\theta)$$

$$\Rightarrow \frac{dr}{d\theta} = a\sin\theta$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{a\sin\theta}{r}$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{\cancel{a}\sin\theta}{\cancel{a}(1-\cos\theta)}$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{2\sin(\theta/2)\cos(\theta/2)}{2\sin^2(\theta/2)}$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \cot(\theta/2) \quad \text{——(2)}$$

but, $\frac{dr}{d\theta} = -r^2 \frac{d\theta}{dr}$

$$\therefore (2) \Rightarrow \frac{1}{r}\left[-r^2 \frac{d\theta}{dr}\right] = \cot(\theta/2)$$

$$\Rightarrow -r\frac{d\theta}{dr} = \cot\left(\frac{\theta}{2}\right)$$

$$\Rightarrow -\frac{1}{r}\,dr = -\frac{1}{\cot\left(\frac{\theta}{2}\right)}\,d\theta$$

$$\Rightarrow \int \frac{1}{r}\,dr = -\int \tan(\theta/2)\,d\theta$$

$$\Rightarrow \log r = \frac{-\log|\sec(\theta/2)|}{1/2} + \log K$$

$$\Rightarrow \log r = -2\log|\sec(\theta/2)| + \log K$$

$$\Rightarrow \log r = \log|\cos^2(\theta/2)| + \log K$$

$$\Rightarrow \log r = \log|K\cos^2(\theta/2)|$$

$$\Rightarrow r = K\cos^2\left[\frac{\theta}{2}\right]$$

$$\Rightarrow r = K\left[\frac{1+\cos\theta}{2}\right]$$

$$\Rightarrow r = \frac{K}{2}(1+\cos\theta)$$

$$\Rightarrow r = b(1+\cos\theta)$$

iii) $r = a(1+\cos\theta)$

$\Rightarrow r = a(1+\cos\theta)$ ——(1)

diff (1) w.r to 'θ'

$$\Rightarrow \frac{dr}{d\theta} = a(0-\sin\theta)$$

$$\Rightarrow \frac{dr}{d\theta} = -a\sin\theta$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = -\frac{a\sin\theta}{r}$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = -\frac{\cancel{a}\sin\theta}{\cancel{a}(1+\cos\theta)}$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = -\frac{2\sin(\theta/2)\cos(\theta/2)}{2\cos^2(\theta/2)}$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = -\tan(\theta/2) \quad \text{---} (2)$$

but, $\frac{dr}{d\theta} = -r^2 \frac{d\theta}{dr}$

$$\therefore (2) \Rightarrow \frac{1}{r}\left[-r^2\frac{d\theta}{dr}\right] = -\tan(\theta/2)$$

$$\Rightarrow -r\frac{d\theta}{dr} = -\tan(\theta/2)$$

$$\Rightarrow -\frac{1}{r}\,dr = -\tan(\theta/2)\,d\theta$$

$$\Rightarrow \int -\frac{1}{r}\,dr = \int -\tan(\theta/2)\,d\theta$$

$$\Rightarrow \log r = \frac{\log|\sec(\theta/2)|}{1/2} + \log K$$

$$\Rightarrow \log r = 2\log|\sec(\theta/2)| + \log K$$

$$\Rightarrow \log r = \log \sec^2(\theta/2) + \log K$$

$$\Rightarrow \log r = \log K\sec^2(\theta/2)$$

$$\Rightarrow r = K\sec^2(\theta/2)$$

iii) $r^n = a^n \sin n\theta$

$\Rightarrow \quad r^n = a^n \sin n\theta$ ——(1)

diff (1) w.r to 'θ'

$\therefore (1) \Rightarrow \cancel{n} r^{n-1} \cdot \frac{dr}{d\theta} = a^n \cos n\theta \cdot \cancel{n}$

$\Rightarrow \frac{r^n}{r} \cdot \frac{dr}{d\theta} = a^n \cos n\theta$

$\Rightarrow \frac{1}{r} \cdot \frac{dr}{d\theta} = \frac{a^n \cos n\theta}{r^n}$

$\Rightarrow \frac{1}{r} \cdot \frac{dr}{d\theta} = \frac{\cancel{a^n} \cos n\theta}{\cancel{a^n} \sin n\theta}$

$\Rightarrow \frac{1}{r} \cdot \frac{dr}{d\theta} = \cot n\theta$ ——(2)

$\therefore$ but $\frac{dr}{d\theta} = -r^2 \frac{d\theta}{dr}$

$\therefore (2) \Rightarrow \frac{1}{r}\left[-r^2 \frac{d\theta}{dr}\right] = \cot n\theta$

$\Rightarrow -r \frac{d\theta}{dr} = \cot n\theta$

$\Rightarrow \int \frac{1}{r}\, dr = -\int \frac{1}{\cot n\theta}\, d\theta$

$\Rightarrow \log r = -\int \tan n\theta \, d\theta$

$\Rightarrow \log r = -\frac{\log|\sec n\theta|}{n} + \log b$

$\Rightarrow n \log r = -\log|\sec n\theta| + n \log b$

$\Rightarrow \log r^n = \log|\cos n\theta| + \log b^n$

$\Rightarrow \log(r^n) = \log(b^n \cos n\theta)$

$\Rightarrow r^n = b^n \cos n\theta$

iv) $\frac{2a}{r} = 1 - \cos\theta$

$\Rightarrow \quad 2a = r(1-\cos\theta)$ ——— (1)

diff w.r to 'θ'

$\Rightarrow \quad 0 = r(a+\sin\theta) + (1-\cos\theta)\frac{dr}{d\theta}$

$\Rightarrow \quad 0 = r\sin\theta + (1-\cos\theta)\frac{dr}{d\theta}$

$\Rightarrow \quad -\frac{r\sin\theta}{1-\cos\theta} = \frac{dr}{d\theta}$

$\Rightarrow \quad \frac{1}{r}\frac{dr}{d\theta} = -\frac{\sin\theta}{1-\cos\theta}$

$\Rightarrow \quad \frac{1}{r}\frac{dr}{d\theta} = -\frac{2\sin(\theta/2)\cos(\theta/2)}{2\sin^2(\theta/2)}$

$\Rightarrow \quad \frac{1}{r}\frac{dr}{d\theta} = -\cot(\theta/2)$ ——— (2)

But, $\frac{dr}{d\theta} = -r^2\frac{d\theta}{dr}$

$\therefore (2) \Rightarrow \frac{1}{r}\left[-r^2\frac{d\theta}{dr}\right] = -\cot(\theta/2)$

$\Rightarrow \quad -r\frac{d\theta}{dr} = -\cot(\theta/2)$

$\Rightarrow \quad \int\frac{1}{r}\frac{d\theta}{dr} = \int\frac{1}{\cot(\theta/2)}d\theta$

$\Rightarrow \quad \log r = \int\tan(\theta/2)\,d\theta$

$\Rightarrow \quad \log r = 2\log|\sec(\theta/2)| + \log K$

$\Rightarrow \quad \log r = \log\sec^2(\theta/2) + \log K$

$\Rightarrow \quad \log r = \log|\sec^2(\theta/2).K|$

$\Rightarrow \quad r = K\sec^2(\theta/2)$

6.

$\Rightarrow$ Given : $r = 2a(\cos\theta + \sin\theta)$ ——(1)

$$\Rightarrow \frac{dr}{d\theta} = 2a(-\sin\theta + \cos\theta) \quad \text{——(2)}$$

$$(2) \div (1) \Rightarrow \frac{dr/d\theta}{r} = \frac{2a(\cos\theta - \sin\theta)}{2a(\cos\theta + \sin\theta)}$$

$$\Rightarrow \frac{1}{r} \cdot \frac{dr}{d\theta} = \frac{1 - \tan\theta}{1 + \tan\theta}$$

$$\Rightarrow \frac{1}{r} \cdot \frac{dr}{d\theta} = \tan\left[\frac{\pi}{4} - \theta\right]$$

but $\dfrac{dr}{d\theta} = -r^2 \dfrac{d\theta}{dr}$

$$\therefore (2) \Rightarrow \frac{1}{r}\left[-r^2 \cdot \frac{d\theta}{dr}\right] = \tan\left[\frac{\pi}{4} - \theta\right]$$

$$\Rightarrow -r \cdot \frac{d\theta}{dr} = \tan\left[\frac{\pi}{4} - \theta\right]$$

$$\Rightarrow \frac{1}{r} \cdot \frac{d\theta}{dr} = \frac{1}{\tan\left[\frac{\pi}{4} - \theta\right]} \cdot d\theta$$

$$\Rightarrow \int \frac{1}{r} \cdot \frac{d\theta}{dr} = \int \cot\left(\frac{\pi}{4} - \theta\right) \cdot d\theta$$

$$\Rightarrow \log r = -\frac{\log|\sec(\pi/4 - \theta)|}{-1} + \log b$$

$$\Rightarrow \log r = \log|\sin(\pi/4 - \theta)| + \log b$$

$$\Rightarrow \log r = \log|b \sin(\pi/4 - \theta)|$$

$$\Rightarrow r = b\sin(\pi/4 - \theta)$$

$$\Rightarrow r = b\left[\sin\pi/4 \cos\theta - \cos\frac{\pi}{4}\sin\theta\right]$$

$$\Rightarrow r = \frac{b}{\sqrt{2}} (\cos\theta - \sin\theta)$$

Applications of Differential Equations

Newton's Law of cooling :-

Let 'T' be the temperature of the body and 'T_0' be the surrounding medium / room temperature at any time 't', then the Newton's law can state that " The rate of change of body temperature is directly proportional to the difference of the temperature of the body and its surrounding medium.

$$\therefore \quad \frac{dT}{dt} \propto (T - T_0)$$

$$\Rightarrow \frac{dT}{dt} = -k(T - T_0)$$

$$\Rightarrow \frac{1}{T - T_0} dT = -k\,dt$$

$$\Rightarrow \int \frac{1}{T - T_0} dT = -kt + c$$

$$\Rightarrow \log_e [T - T_0] = -kt + c$$

$$\Rightarrow T - T_0 = e^{-kt + c}$$

$$\Rightarrow T - T_0 = e^{-kt} \cdot e^{c}$$

$$\Rightarrow T = T_0 + e^{c} \cdot e^{-kt}$$

$$\Rightarrow \boxed{T = T_0 + \lambda e^{-kt}}$$

1. A body originally $80^\circ c$ cools down to $60^\circ c$ in 20 min the temp of the air being $40^\circ c$, what will be the temperature of the body after 40 min from the original.

$\Rightarrow$ Given,

Air temperature $[T_0] = 40^\circ c$

WKT, $T = T_0 + \lambda e^{-Kt}$ ——(1)

$\Rightarrow T = 40 + \lambda e^{-Kt}$ ——(2)

Also given,

The body given temp $[T] = 80^\circ c$ at $t=0$

$\therefore (2) \Rightarrow 80 = 40 + \lambda e^{-Kt}$

$\Rightarrow 80 - 40 = \lambda$

$\Rightarrow \lambda = 40$

$\therefore (2) \Rightarrow T = 40 + 40\, e^{-Kt}$ ——(3)

Also given, The body temp is reduced to $60^\circ c$ after 20 min.

$\therefore T = 60^\circ c$

$(3) \Rightarrow 60 = 40 + 40\, e^{-20K}$

$\Rightarrow 40\, e^{-20K} = 20$

$\Rightarrow e^{-20K} = \frac{20}{40}$

$\Rightarrow e^{-20K} = 0.5$

$\Rightarrow -20K = \log_e(0.5)$

$\Rightarrow -20K = -0.69314$

$\Rightarrow K = \frac{0.69314}{20} = 0.03465$

$\therefore (2) \Rightarrow \quad T = 40 + 40e^{-(0.03465)t}$

$\therefore$ At, $t = 40$

$\Rightarrow \quad T = 40 + 40e^{-(0.03465)(40)}$

$\Rightarrow \quad T = 40 + 40e^{-1.386}$

$\Rightarrow \quad T = 40 + 40(0.25)$

$\Rightarrow \quad T = 40 + 10$

$\Rightarrow \quad \boxed{T = 50^{\circ}c}$

2. If the temperature of air is $30^{\circ}c$ and a metal ball cools from $100^{\circ}c$ to $70^{\circ}c$ in 15 min. Find how long will it take for the metal ball to reach the temperature of $40^{\circ}c$?

$\Rightarrow$ Given,

Air temperature $[T_0] = 30^{\circ}c$

WKT, $\quad T = T_0 + \lambda e^{-Kt}$ —— (1)

$\Rightarrow \quad T = 30 + \lambda e^{-Kt}$ —— (2)

Given,

Temperature of metal ball $[T] = 100^{\circ}c$ at $t = 0$

$\therefore$ eqn (2) :- $\Rightarrow \quad 100 = 30 + \lambda e^{0}$

$\Rightarrow \quad 100 - 30 = \lambda$

$\Rightarrow \quad \lambda = 70$

$(2) \Rightarrow \quad T = 30 + \lambda e^{-Kt}$

$T = 30 + 70e^{-Kt}$ —— (3)

Also given, the metal ball temp reduced to $70^{\circ}c$ after 15 min.

$(3) \Rightarrow \quad 70 = 30 + 70e^{-15K}$

$$\Rightarrow \quad 70e^{-15K} = 40$$

$$\Rightarrow \quad e^{-15K} = \frac{40}{70}$$

$$\Rightarrow \quad e^{-15K} = 0.5714$$

$$\Rightarrow \quad -15K = \log_e(0.5714)$$

$$\Rightarrow \quad -15K = -0.5596$$

$$\Rightarrow \quad K = \frac{0.5596}{15}$$

$$\Rightarrow \quad K = 0.0373$$

$$\therefore (3) \Rightarrow \quad T = 30 + 70e^{-(0.0373)t} \quad \text{——} (4)$$

At , $T = 40^\circ c$

$$(4) \Rightarrow \quad 40 = 30 + 70e^{-(0.0373)t}$$

$$\Rightarrow \quad 70e^{-(0.0373)t} = 10$$

$$\Rightarrow \quad e^{-(0.0373)t} = \frac{10}{70}$$

$$\Rightarrow \quad e^{-(0.0373)t} = 0.1428$$

$$\Rightarrow \quad -(0.0373)t = \log_e(0.1428)$$

$$\Rightarrow \quad -(0.0373)t = -1.9463$$

$$\Rightarrow \quad t = \frac{1.9463}{0.0373}$$

$$\Rightarrow \quad t = 52.18 \text{ min}$$

3. Water at temperature 10°c takes 5 min to warm upto 20°c at room temperature of 40°c. Find the temperature of water after 20 min.

⇒ Given,

room temperature $[T_0] = 40°c$

WKT, The netwon's law of warming.

$$T = T_0 + \lambda e^{Kt}$$

$$\Rightarrow T = 40 + \lambda e^{Kt} \quad \text{——(1)}$$

Given, that the water is at 10°c at $t=0$

$$(1) \Rightarrow T = 40 + \lambda e^{0}$$

$$\Rightarrow 10 = 40 + \lambda$$

$$\Rightarrow \lambda = -30$$

$$\therefore (1) \Rightarrow T = 40 - 30e^{-Kt} \quad \text{——(2)}$$

Also given, the water was heated to 20°c after 5 min

$$\therefore (2) \Rightarrow 20 = 40 - 30e^{5K}$$

$$\Rightarrow -30e^{5K} = -20$$

$$\Rightarrow e^{5K} = \frac{20}{30}$$

$$\Rightarrow e^{5K} = 0.6666$$

$$\Rightarrow 5K = \log_e(0.6666)$$

$$\Rightarrow 5K = -0.4055$$

$$\Rightarrow K = -0.0811$$

$$(2) \Rightarrow T = 40 - 30e^{-(0.0811)t} \quad \text{——(3)}$$

$\therefore$ At $t = 20$ min

$(3) \Rightarrow \quad T = 40 - 30\, e^{-(0.0811)(20)}$

$\Rightarrow \quad T = 40 - 30 e^{-1.622}$

$\Rightarrow \quad T = 40 - 30\,(0.1975)$

$\Rightarrow \quad T = 40 - 5.9250$

$\Rightarrow \quad \boxed{T = 34.07^\circ C}$

4. A body is heated at 110°c and placed in the air at 10°c after an hour its temperature becomes 60°c. How much additional time is required to cool 30°c at t=0.

$\Rightarrow$ Given,

$T_0 = 10^\circ c$

WKT,

$T = T_0 + \lambda e^{-kt}$

$\Rightarrow \quad T = 10 + \lambda e^{-kt}$ —— (1)

given the water is cooled to 60°c at t = 1 hour = 60 min

$\Rightarrow \quad 60 + 10 + \lambda e^{-60k}$

also given :-

$\therefore \quad T = 110^\circ c \quad$ at $\quad t = 0$ min

$\therefore (2) \Rightarrow \quad 110 = 10 + \lambda e$

$\lambda = 110 - 10$

$\lambda = 100$

$\therefore (2) \Rightarrow \quad T = 10 + 100\, e^{-kt}$ —— (3)

also given, $\quad T = 60^\circ c \quad$ at $\quad t = 60$ min

$(3) \Rightarrow \quad 60 - 10 = 100\, e^{-60k}$

$\Rightarrow \quad \frac{50}{100} = e^{-60k}$

$\Rightarrow \quad e^{-60K} = 0.5$

$\Rightarrow \quad -60K = \log_e(0.5)$

$\Rightarrow \quad K = \frac{0.69}{60}$

$\Rightarrow \quad K = 0.0115$

$\therefore (3) \Rightarrow \quad T = 10 + 100e^{-0.0115t}$

At, $T = 30^{\circ}$

$\Rightarrow \quad 30 = 10 + 100e^{-(0.0115)t}$

$\Rightarrow \quad 30 - 10 = 100e^{-(0.0115)t}$

$\Rightarrow \quad e^{-(0.0115)t} = \frac{20}{100}$

$\Rightarrow \quad -(0.0115)t = -0.2$

$\Rightarrow \quad t = \frac{0.2}{0.0115}$

$\Rightarrow \quad t = 17.39$ min

5. A copper ball at $80^{\circ}c$ at cools down to $60^{\circ}c$ in 20 min. If the temperature of the room is being $40^{\circ}c$, What will be the temperature of the ball after 40 min from the original.

$\Rightarrow$ Given,

The room temperature, $T_0 = 40^{\circ}c$

WKT,

$$T = T_0 + \lambda e^{-kt} \quad \text{——(1)}$$

$$\Rightarrow T = 40 + \lambda e^{-kt} \quad \text{——(2)}$$

also given, $T = 80^{\circ}c$ at $t = 0$

$(2) \Rightarrow \quad 80 = 40 \lambda e^{0}$

$\Rightarrow \quad \lambda = 40$

$\therefore (2) \Rightarrow \; T = 40 + 40e^{-kt}$ ——— (3)

and given, $T = 60^{\circ}c$ at $t = 20\,min$

$\therefore (3) \Rightarrow \; 60 = 40 + 40e^{-20k}$

$\Rightarrow \; 40e^{-20k} = 20$

$\Rightarrow \; e^{-20k} = \frac{20}{40}$

$\Rightarrow \; e^{-20k} = 0.5$

$\Rightarrow \; -20k = \log_e(0.5)$

$\Rightarrow \; -20k = -0.6931$

$\Rightarrow \; k = \frac{0.6931}{20}$

$\Rightarrow \; k = 0.03465$

$\therefore (3) \Rightarrow T = 40 + 40e^{-0.03465t}$ ——— (4)

$\therefore$ At $t = 40$ min

$\therefore \; T = 40 + 40e^{(-0.03465)(40)}$

$\Rightarrow T = 40 + 40e^{(-1.386)}$

$\Rightarrow T = 40 + 40(0.250)$

$\Rightarrow T = 40 + 10$

$\Rightarrow \boxed{T = 50^{\circ}c}$

6. A body is at $25^{\circ}c$, whose from $100^{\circ}c$ to $75^{\circ}c$ in one min. Find the temperature of the body at 3min.

$\Rightarrow$ Given,

$$T_0 = 25^\circ C$$

WKT,

$$T = T_0 + \lambda e^{-Kt} \quad \text{——(1)}$$

$$\Rightarrow T = 25 + \lambda e^{-Kt} \quad \text{——(2)}$$

at $T = 100$, $t = 0$

$$\therefore (2) \Rightarrow 100 = 25 + \lambda e^{0}$$

$$\Rightarrow 75 = \lambda$$

$$(2) \Rightarrow T = 25 + 75 e^{-Kt} \quad \text{——(3)}$$

also given, at $T = 75^\circ$, $t = 1 min$

$$(3) \Rightarrow 75 = 25 + 75 e^{-K(1)}$$

$$\Rightarrow 75 - 25 = 75 e^{-K}$$

$$\Rightarrow 50 = 75 e^{-K}$$

$$\Rightarrow e^{-K} = \frac{50}{75}$$

$$\Rightarrow e^{-K} = 0.6666$$

$$\Rightarrow -K = \log_e (0.6666)$$

$$\Rightarrow -K = -0.4055$$

$$\Rightarrow K = 0.4055$$

$$\therefore (3) \Rightarrow T = 25 + 75 e^{(-0.4055)t} \quad \text{——(4)}$$

at $t = 3 min$

$$\therefore (4) \Rightarrow T = 25 + 75 e^{(-0.4055)(3)}$$

$$= 25 + 75 e^{-1.2165}$$

$$= 25 + 75 (0.2962)$$

$$= 25 + 22.215$$

$$= 47.215$$

Flow of Electricity

Non - Linear differential Equations :-

Solvable for p method

1. Let $y = f(x)$ be a function and let $p = dy/dx$, then the general non-linear differential equation can be defined as

$$A_0 p^n + A_1 p^{n-1} + A_2 p^{n-2} + \cdots A_n = 0 \quad \text{—— (1)}$$

where $A_0, A_1, A_2, \ldots A_n$ are the functions of the variables x & y and also eqn (1) is a polynomial of degree n, it follows as

2. $(1) \Rightarrow [p - f_1(x,y)][p - f_2(x,y)][p - f_3(x,y)] \cdots [p - f_n(x,y)] = 0$

$$\Rightarrow p - f_1(x,y) = 0, \; p - f_2(x,y) = 0, \; p - f_3(x,y) = 0 \cdots p - f_n(x,y) = 0$$

$$\Rightarrow p = f_1(x,y), \; p = f_2(x,y), \; p = f_3(x,y) \cdots p = f_n(x,y)$$

3. Solve the above equations and get the solutions $F_1(x,y,C_1) = 0$, $F_2(x,y,C_2) = 0$, $F_3(x,y,C_3) = 0 \cdots F_n(x,y,C_n) = 0$

$\therefore$ The complete solution of given non-linear differential equation can be written as,

$$f_1(x,y,C_1) \cdot f_2(x,y,C_2) \cdot f_3(x,y,C_3) \cdots f_n(x,y,C_n) = 0$$

1. Solve $xyp^2 - (x^2+y^2)p + xy = 0$

$\Rightarrow$ Given :- $xyp^2 - (x^2+y^2)p + xy = 0$ ——— (1)

$\Rightarrow xyp^2 - x^2p - y^2p + xy = 0$

$\Rightarrow xp(yp - x) - y(yp - x) = 0$

$\Rightarrow (px - y)(py - x) = 0$

$\Rightarrow px - y = 0 \quad , \quad py - x = 0$

$\Rightarrow px = y \quad , \quad py = x$

$\Rightarrow p = \frac{y}{x} \quad , \quad p = \frac{x}{y}$

case (1) : $p = \frac{y}{x}$

$\Rightarrow \frac{dy}{dx} = \frac{y}{x}$

$\Rightarrow \int \frac{1}{y} \cdot dy = \int \frac{1}{x} \cdot dx$

$\Rightarrow \log y = \log x + \log c_1$

$\Rightarrow \log y = \log (c_1 x)$

$\Rightarrow y = c_1 x$

$\Rightarrow y - c_1 x = 0$

case (2) : $p = x/y$

$\Rightarrow \frac{dy}{dx} = \frac{x}{y}$

$\Rightarrow \int y \cdot dy = \int x \cdot dx$

$\Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + c_2$

$$\Rightarrow \quad y^2 = x^2 + 2c_2$$

$$\Rightarrow \quad y^2 - x^2 - 2c_2 = 0$$

$\therefore$ The complete solution of equation (1) is

$$(y - c_1 x)(y^2 - x^2 - 2c_2) = 0$$

2. Solve $\dfrac{dy}{dx} - \dfrac{dx}{dy} = \dfrac{x}{y} - \dfrac{y}{x}$

$\Rightarrow$ Given, $\dfrac{dy}{dx} - \dfrac{dx}{dy} = \dfrac{x}{y} - \dfrac{y}{x}$ ——(1)

but $\dfrac{dy}{dx} = p$

$$\therefore (1) \Rightarrow p - \frac{1}{p} = \frac{x^2 - y^2}{xy}$$

$$\Rightarrow \frac{p^2 - 1}{p} = \frac{x^2 - y^2}{xy}$$

$$\Rightarrow (p^2 - 1)\, xy = (x^2 - y^2)\, p$$

$$\Rightarrow xyp^2 - (x^2 - y^2)\, p - xy = 0$$

$$\Rightarrow xyp^2 - x^2 p + y^2 p - xy = 0$$

$$\Rightarrow xp(yp - x) + y(yp - x) = 0$$

$$\Rightarrow (yp - x)(xp + y) = 0$$

$$\Rightarrow yp - x = 0 \quad , \quad xp + y = 0$$

$$\Rightarrow yp = x \quad , \quad xp = -y$$

$$\Rightarrow p = \frac{x}{y} \quad , \quad p = -\frac{y}{x}$$

Case (1) :-

$$p = \frac{x}{y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x}{y}$$

$$\Rightarrow \int y \cdot dy = \int x \cdot dx$$

$$\Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + c_1$$

$$\Rightarrow y^2 = x^2 + 2c_1$$

$$\Rightarrow y^2 - x^2 - 2c_1 = 0$$

Case (2) :

$$p = -y/x$$

$$\Rightarrow \frac{dy}{dx} = -\frac{y}{x}$$

$$\Rightarrow \int \frac{1}{y}\, dy = -\int \frac{1}{x}\, dx$$

$$\Rightarrow \log y = -\log x + \log c_2$$

$$\Rightarrow \log y = \log \left[c_2/x \right]$$

$$\Rightarrow y = c_2/x$$

$$\Rightarrow xy = c_2$$

$$\Rightarrow xy - c_2 = 0$$

$\therefore$ The solution is $(y^2 - x^2 - c_1)(xy - c_2) = 0$

3. Solve, $p^2 + 2py\cot x - y^2 = 0$

$\Rightarrow$ Given,

$$p^2 + 2py\cot x - y^2 = 0$$

$\therefore a = 1, \quad b = 2y\cot x, \quad c = -y^2$

$$\therefore\ p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$p = \frac{-2y\cot x + \sqrt{(2y\cot x)^2 - 4(1)(-y^2)}}{2(1)}$$

$$p = \frac{-2y\cot x \pm \sqrt{(4y^2\cot x)^2 + 4y^2}}{2}$$

$$p = \frac{-2y\cot x \pm 2y\sqrt{\cot^2 x + 1}}{2}$$

$$p = -y\cot x \pm y\,\mathrm{cosec}\,x$$

case(1) :-

$$p = -y\cot x + y\,\mathrm{cosec}\,x$$

$$\Rightarrow \frac{dy}{dx} = y\left[\mathrm{cosec}\,x - \cot x\right]$$

$$\Rightarrow \frac{1}{y}\cdot dy = \left[-\cot x + \mathrm{cosec}\,x\right] dx$$

$$\Rightarrow \frac{1}{y}\cdot dy = \left[\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right] dx$$

$$\Rightarrow \frac{1}{y}\cdot dy = \left[\frac{1-\cos x}{\sin x}\right] dx$$

$$\Rightarrow \frac{1}{y}\cdot dy = \frac{2\sin^2 x}{2\sin(x/2)\cos(x/2)}\,dx$$

$$\Rightarrow \int \frac{1}{y}\cdot dy = \int \tan(x/2)\,dx$$

$$\Rightarrow \log y = \frac{\log|\sec(x/2)|}{1/2} + \log c_1$$

$$\Rightarrow \log y = 2\log|\sec(x/2)| + \log c_1$$

$$\Rightarrow \log y = \log|c_1 \sec^2(x/2)|$$

$$\Rightarrow y = c_1 \sec^2(x/2)$$

$$\Rightarrow y = \frac{c_1}{\cos^2(x/2)}$$

$$\Rightarrow y = \frac{c_1}{\left(\frac{1+\cos x}{2}\right)}$$

$$\Rightarrow y = \frac{2c_1}{1+\cos x}$$

$$\Rightarrow y(1+\cos x) = 2c_1$$

$$\Rightarrow y(1+\cos x) - 2c_1 = 0$$

Case (2):-

$$\frac{1}{y}\,dy = -(\cot x + \operatorname{cosec} x)$$

$$\Rightarrow \frac{1}{y}\,dy = -\left[\frac{\cos x}{\sin x} + \frac{1}{\sin x}\right] dx$$

$$\Rightarrow \frac{1}{y}\,dy = -\frac{(1+\cos x)}{\sin x}\,dx$$

$$\Rightarrow \frac{1}{y}\,dy = \frac{-2\cos^2 x}{2\sin(x/2)\cos(x/2)}\,dx$$

$$\Rightarrow \int \frac{1}{y}\cdot dy = -\int \cot(x/2)\,dx$$

$$\Rightarrow \log y = \frac{-\log|\sin(x/2)|}{1/2} + \log c_2$$

$$\Rightarrow \log y = -2\log|\sin(x/2)| + \log C_2$$

$$\Rightarrow \log y = -\log|\sin^2(x/2)| + \log C_2$$

$$\Rightarrow \log y = \log\left[\frac{C_2}{\sin^2(x/2)}\right]$$

$$\Rightarrow y = \frac{C_2}{\sin^2(x/2)}$$

$$\Rightarrow y = \frac{C_2}{\left(\frac{1-\cos x}{2}\right)}$$

$$\Rightarrow y = \frac{2C_2}{1-\cos x}$$

$$\Rightarrow y(1-\cos x) = 2C_2$$

$$\Rightarrow y(1-\cos x) - 2C_2 = 0$$

4. Solve $p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0$

$$\Rightarrow p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0$$

$$\Rightarrow p[p^2 + 2xp - y^2p - 2xy^2] = 0$$

$$\Rightarrow p = 0,\ p^2 + 2xp - y^2p - 2xy^2 = 0$$

$$\Rightarrow p = 0,\ p[p+2x] - y^2[p+2x] = 0$$

$$\Rightarrow p = 0,\ (p-y^2)(p+2x) = 0$$

$$\Rightarrow p = 0,\ p - y^2 = 0,\ p + 2x = 0$$

$$\Rightarrow p = 0,\ p = y^2,\ p = -2x$$

case (1) :- $p = 0$

$$\Rightarrow \frac{dy}{dx} = 0$$

$$\Rightarrow \int dy = \int 0 \, dx$$

$$\Rightarrow y = C_1$$

$$\Rightarrow y - C_1 = 0$$

case (2) :-

$$p = y^2$$

$$\Rightarrow \frac{dy}{dx} = y^2$$

$$\Rightarrow \int \frac{1}{y^2} dy = \int dx$$

$$\Rightarrow -\frac{1}{y} = x + C_2$$

$$\Rightarrow x + \frac{1}{y} + C_2 = 0$$

case (3) :-

$$p = -2x$$

$$\Rightarrow \frac{dy}{dx} = -2x$$

$$\Rightarrow \int dy = -2 \int x \, dx$$

$$\Rightarrow y = -2\frac{x^2}{2} + C_3$$

$$\Rightarrow y = -x^2 + C_3$$

$$\Rightarrow y + x^3 - C_3 = 0$$

$\therefore$ The solution is, $(y - C_1)(x + 1/y + C_2)(y + x^2 - C_3) = 0$

5. Solve, $y\left[\frac{dy}{dx}\right]^2 + (x-y)\frac{dy}{dx} - x = 0$

$\Rightarrow \quad y\left[\frac{dy}{dx}\right]^2 + (x-y)\frac{dy}{dx} - x = 0$

$\Rightarrow \quad yp^2 + (x-y)p - x = 0$

$\Rightarrow \quad yp^2 - xp - yp - x = 0$

$\Rightarrow \quad p(yp+x) - 1(yp+x) = 0$

$\Rightarrow \quad (p-1)(yp+x) = 0$

$\Rightarrow \quad p-1 = 0 \quad , \quad yp + x = 0$

$\Rightarrow \quad p = 1 \quad , \quad yp = -x$

$\Rightarrow \quad p = 1 \quad , \quad p = -\frac{x}{y}$

case (1) :- $p = 1$

$\Rightarrow \frac{dy}{dx} = 1$

$\Rightarrow \int dy = \int dx$

$\Rightarrow y = x + c_1$

$\Rightarrow y - x - c_1 = 0$

case (2) :- $p = -\frac{x}{y}$

$\Rightarrow \frac{dy}{dx} = -\frac{x}{y}$

$\Rightarrow y\,dy = -x\,dx$

$\Rightarrow \int y\,dy = -\int x\,dx$

$\Rightarrow \frac{y^2}{2} = -\frac{x^2}{2} + c_2$

$\Rightarrow y^2 + x^2 - 2c_2 = 0$

Claurit's Equation:-

1. Write the given non-linear differential eqn to be in the form of $y = px + f(p)$, it is called the Claurit's Equation.

2. To get the general solution for claurit's equation, substitute $p = c$ in the claurit's equation, we get

$$y = cx + f(c) \quad \text{—— (1)}$$

3. To get similar solution, diff (1) partially w.r to 'c' and hence substitute the 'c' value in the general solution.

1. Solve the claurit's equation $y = px + \frac{a}{p}$

$\Rightarrow$ Given :- $y = px + \frac{a}{p}$ —— (1)

$\therefore$ eqn(1) is in claurin's form

The solution is

$$y = cx + \frac{a}{c} \quad \text{—— (2)}$$

diff (2) partially w.r to 'c'

$$\therefore (2) \Rightarrow \quad 0 = x - \frac{a}{c^2}$$

$$\Rightarrow \quad \frac{a}{c^2} = x$$

$$\Rightarrow \quad c^2 = \frac{a}{x}$$

$$\Rightarrow \quad c = \sqrt{\frac{a}{x}}$$

$$\therefore (2) \Rightarrow y = \sqrt{\frac{a}{x}} \cdot x + \frac{a}{\sqrt{\frac{a}{x}}}$$

$$\Rightarrow y = \frac{\sqrt{a}}{\sqrt{x}} \sqrt{x}\sqrt{x} + \frac{\sqrt{a}\sqrt{a}}{\sqrt{a}/\sqrt{x}}$$

$$\Rightarrow y = \sqrt{a}\sqrt{x} + \sqrt{a}\sqrt{x}$$

$$\Rightarrow y = 2\sqrt{ax}$$

2. Show that equation $xp^2 + px - py - y + 1 = 0$ is a clairit's equation and hence find its general and singular solution.

$\Rightarrow$ Given,

$$xp^2 + px - py + 1 = 0$$

$$\Rightarrow xp^2 + px - py - y = -1$$

$$\Rightarrow xp(p+1) - y(p+1) = -1$$

$$\Rightarrow (xp - y)(p+1) = -1$$

$$\Rightarrow (xp - y) = \frac{-1}{p+1}$$

$$\Rightarrow -y = -px - \frac{1}{p+1}$$

$$\Rightarrow y = px + \left[\frac{1}{p+1}\right] \quad \text{——(1)}$$

$\therefore$ eqn (1) is in clairit's form

$\therefore$ The general solution is

$$y = cx + \frac{1}{c+1} \quad \text{——(2)}$$

diff (2) w.r to 'c' partially

$\therefore$ (2) $\Rightarrow$ $0 = x - \frac{1}{(c+1)^2}$

$\Rightarrow \frac{1}{(c+1)^2} = x$

$\Rightarrow (c+1)^2 = \frac{1}{x}$

$\Rightarrow c+1 = \frac{1}{\sqrt{x}}$

$\Rightarrow c = \frac{1}{\sqrt{x}} - 1$

$\therefore$ The singular solution is

$y = \left(\frac{1}{\sqrt{x}} - 1\right)x + \frac{1}{1/\sqrt{x}}$

$\Rightarrow y = \frac{x}{\sqrt{x}} - x + \sqrt{x}$

$\Rightarrow y = \sqrt{x} - x + \sqrt{x}$

$\Rightarrow y = 2\sqrt{x} - x$

3. Show that the equation $xp^3 - yp^2 + 1 = 0$ is the clawrit's eqn and hence find its solution.

$\Rightarrow$ Given, $xp^3 - yp^2 + 1 = 0$

$\Rightarrow xp^3 - yp^2 = -1$

$\Rightarrow p^2(px - y) = -1$

$\Rightarrow px - y = \frac{-1}{p^2}$

$\Rightarrow y = px + \frac{1}{p^2}$ ——— (1)

$\therefore$ Eqn (1) is in clairit's form

$\therefore$ The general solution

(1) $\Rightarrow$ $y = cx + \frac{1}{c^2}$ ——(2)

diff (2) w.r to 'c' partially

(2) $\Rightarrow$ $0 = x - \frac{1}{c}$

$\Rightarrow$ $\frac{1}{c} = x$

$\Rightarrow$ $c = 1/x$

$\therefore$ The singular soln is

$$y = \left(\frac{1}{x}\right)x + x^2$$

$\Rightarrow$ $y = 1 + x^2$

$\Rightarrow$ $x^2 - y + 1 = 0$

4. Reduce the equation $(px-y)(py-x) = 2p$ to the clairit's form, taking the substitution $X = x^2$ and $Y = y^2$ and hence find its solution.

$\Rightarrow$ Given:-

$(px-y)(py-x) = 2p$ ——(1)

$X = x^2$ ——(2)

$Y = y^2$ ——(3)

$\therefore$ $\frac{dX}{dx} = 2x$, $\frac{dY}{dy} = 2y$

$\Rightarrow$ $dX = 2x\,dx$, $dY = 2y\,dy$

$\Rightarrow$ $\frac{dY}{dX} = \frac{\cancel{2}y\,dy}{\cancel{2}x\,dx}$

$$\Rightarrow \frac{dY}{dX} = \frac{y}{x}\left[\frac{dy}{dx}\right]$$

$$\Rightarrow P = \frac{\sqrt{Y}}{\sqrt{X}}\, p$$

$$\Rightarrow p = \frac{\sqrt{X}}{\sqrt{Y}}\, P$$

$$\therefore (1) \Rightarrow \left[\frac{\sqrt{X}}{\sqrt{Y}} P\sqrt{X} - \sqrt{Y}\right]\left[\frac{\sqrt{X}}{\sqrt{Y}} P.\sqrt{Y} + \sqrt{X}\right] = 2\frac{\sqrt{X}}{\sqrt{Y}} P$$

$$\Rightarrow \left[\frac{PX}{\sqrt{Y}} - \sqrt{Y}\right]\sqrt{X}\,(P+1) = 2\frac{\sqrt{X}}{\sqrt{Y}} P$$

$$\Rightarrow \left[\frac{PX - Y}{\cancel{\sqrt{Y}}}\right]\cancel{\sqrt{X}}\,(P+1) = 2\frac{\cancel{\sqrt{X}}}{\cancel{\sqrt{Y}}} P$$

$$\Rightarrow (PX - Y)(P+1) = 2P$$

$$\Rightarrow PX - Y = \frac{2P}{P+1}$$

$$\Rightarrow -Y = -PX + \frac{2P}{P+1}$$

$$\Rightarrow Y = PX + \frac{2P}{P+1}$$

$\therefore$ The solution is,

$$Y = CX - \frac{2C}{C+1}$$

$$\Rightarrow y^2 = cx^2 - \frac{2c}{c+1}$$

5. Reduce the equation to the clairit's form $x^2(y-px) = p^2 y$ taking the substitutions $X = x^2$, $Y = y^2$ and hence find the general and singular solution?

$\Rightarrow$ $$x^2(y-px) = p^2 y \quad \text{——(1)}$$

$$X = x^2 \quad \text{——(2)}$$

$$Y = y^2 \quad \text{——(3)}$$

$$\therefore \frac{dX}{dx} = 2x \qquad \frac{dY}{dy} = 2y$$

$$\Rightarrow dX = 2x\,dx \quad , \quad dY = 2y\,dy$$

$$\therefore \frac{dY}{dX} = \frac{y}{x}\left[\frac{dy}{dx}\right]$$

$$\Rightarrow P = \frac{\sqrt{Y}}{\sqrt{X}}\, p$$

$$\Rightarrow p = \frac{\sqrt{X}}{\sqrt{Y}}\, P$$

$$(1) \Rightarrow X\left[\sqrt{Y} - \frac{\sqrt{X}}{\sqrt{Y}} P.\sqrt{X}\right] = \left[\frac{\sqrt{X}}{\sqrt{Y}} P\right]^2 \sqrt{Y}$$

$$\Rightarrow \cancel{X}\left[\frac{Y - PX}{\sqrt{Y}}\right] = \frac{\cancel{X}}{Y} P^2 \sqrt{Y}$$

$$\Rightarrow Y - PX = \frac{P^2 \cancel{Y}}{\cancel{Y}}$$

$$\Rightarrow Y - PX = P^2$$

$$\Rightarrow Y = PX + P^2$$

$\therefore$ The soln is

$$Y = CX + C^2 \quad \text{——(4)}$$

$$\Rightarrow \quad y^2 = cx^2 + c^2 \text{ ——— } (5)$$

diff w.r to 'c' partially

$$(5) \Rightarrow \quad 0 = x^2 + 2c$$

$$\Rightarrow \quad 2c = -x^2$$

$$\Rightarrow \quad c = -\frac{x^2}{2}$$

$$(5) \Rightarrow \quad y^2 = \left[-\frac{x^2}{2}\right][x^2] + \left[-\frac{x^2}{2}\right]^2$$

$$\Rightarrow \quad y^2 = \frac{-x^4}{2} + \frac{x^4}{4}$$

$$\Rightarrow \quad y^2 = -\frac{x^4}{4}$$

$$\Rightarrow \quad x^4 + 4y^2 = 0$$

6. Find the general soln of the equation $(px-y)(py+x) = a^2p$ by reducing into clairit's form by taking the substitution $X = x^2$, $Y = y^2$

$$\Rightarrow \quad (px-y)(py+x) = a^2p \text{ ——— } (1)$$

$$X = x^2 \text{ ——— } (2)$$

$$Y = y^2 \text{ ——— } (3)$$

$$\therefore \frac{dX}{dx} = 2x \quad , \quad \frac{dY}{dy} = 2y$$

$$\Rightarrow \quad dX = 2x\,dx \quad , \quad dY = 2y\,dy$$

$$\therefore \quad \frac{dY}{dX} = \frac{y}{x}\left[\frac{dy}{dx}\right]$$

$$\Rightarrow \quad P = \frac{\sqrt{Y}}{\sqrt{X}}\,p \qquad \Rightarrow \quad p = \frac{\sqrt{X}}{\sqrt{Y}}\,P$$

$$(1) \Rightarrow \left[\frac{\sqrt{x}}{\sqrt{y}}\, p\sqrt{x} - \sqrt{y}\right]\left[\frac{\sqrt{x}}{\sqrt{y}}\, p\sqrt{y} + \sqrt{x}\right] = a^2 \frac{\sqrt{x}}{\sqrt{y}}\, p$$

$$\Rightarrow \left[\frac{px}{\sqrt{y}} - \sqrt{y}\right]\left[\sqrt{x}\,[p+1]\right] = a^2 \frac{\sqrt{x}}{\sqrt{y}}\, p$$

$$\Rightarrow \left[\frac{px - y}{\cancel{\sqrt{y}}}\right] \cancel{\sqrt{x}}\,(p+1) = a^2 \frac{\cancel{\sqrt{x}}}{\cancel{\sqrt{y}}}\, p$$

$$\Rightarrow (px - y)(p+1) = a^2 p$$

$$\Rightarrow px - y = \frac{a^2 p}{p+1}$$

$$\Rightarrow -y = -px + \frac{a^2 p}{p+1}$$

$$\Rightarrow y = px + \frac{a^2 p}{p+1}$$

$\therefore$ The solution is;

$$Y = \frac{cX - a^2 c}{c+1}$$

$$\Rightarrow y^2 = cx^2 - \frac{a^2 c}{c+1}$$